Chapter 1

Differential Equations: Basic Concepts

Equations containing derivatives of a dependent variable with respect to independent variables are called differential equations (DEs). For example, the equation

$$y' - y = x^2$$

is a differential equation. In this equation x is the independent variable and $y = y(x)$ is the dependent variable. In the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x^2 + y^2,$$

x and y are the independent variables and $u = u(x, y)$ is the dependent variable.

Differential equations arise in a variety of subject areas, including not only the physical sciences but also such diverse fields as economics, medicine, psychology, and operations research. For example, in banking practice, if $P(t)$ is the number of dollars in a saving bank account that pays a yearly interest rate of $r\%$ compounded continuously, then P satisfies the differential equation

$$\frac{dP}{dt} = \frac{r}{100}P, \qquad t \text{ in years.}$$

Some useful ways of classifying differential equations will be described here.

(1) Order: The order of a differential equation is the order of the highest derivative that appears, on the dependent variable with respect to the independent variables, in the equation. Thus, the equation

$y' = f(x, y)$ is a first-order DE, and $y'' = f(x, y, y')$ is a second-order DE. More generally, we can express an nth-order DE with x independent and y dependent as

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}),$$

or

$$F(x, y, y', \dots, y^{(n)}) = 0.$$

For example, $y''' - 4y'' + 6y^7 = x^{10}$ is a 3rd-order DE involving x as the dependent variable and $y = y(x)$ as the dependent variable.

(2) Linear and Nonlinear Equations: The DE

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is called linear if the dependent variable y and its derivatives appear in additive combinations of their first powers. Thus a DE is linear if it has the format

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = F(x).$$

An equation that is not linear is called nonlinear equation. For example,

$$y'' + x^2y' - (\ln x)\, y = x^{10}$$

is a linear equation, whereas

$$y''' + 3y'' - y' + y^2 = x^3$$

is nonlinear 3rd-order DE because of the y^2 term.

(3) Ordinary and Partial Differential Equations: A differential equation involving only ordinary derivatives with respect to a single independent variable is called an ordinary differential equation (ODE). A differential equation involving partial derivatives with respect to more than one independent variable is a partial differential equation (PDE). Thus, the equation

$$y'' + 3y' - 4y = 0$$

is an ODE and the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is a PDE.

Definition 1.1 A function $\phi(x)$ such that $\phi, \phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)} = F(x, \phi, \phi', \ldots, \phi^{(n-1)})$$

for every x in an interval I is called an explicit solution to the DE

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$$

on I.

Example 1.1 Show that $y(x) = \sin x$ is an explicit solution to the ODE

$$y'' + y = 0.$$

Solution: The functions $y(x) = \sin x$, $y'(x) = \cos x$, and $y''(x) = -\sin x$ are defined for all $x \in (-\infty, \infty)$. Substituting in the DE gives

$$y'' + y = -\sin x + \sin x = 0.$$

Thus $y(x) = \sin x$ is a solution of the DE for all $x \in (-\infty, \infty)$.

Example 1.2 Determine the values of r for which the DE has solution of the form $y = x^r$ for $x > 0$,

$$x^2 y'' - 4xy' + 4y = 0.$$

Solution: The functions $y(x) = x^r$, $y'(x) = rx^{r-1}$, and $y''(x) = r(r-1)x^{r-2}$ are defined for $x > 0$. On substituting in the DE we obtain

$$\begin{aligned} x^2 y'' - 4xy' + 4y &= 0, \\ x^2 r(r-1)x^{r-2} - 4xrx^{r-1} + 4x^r &= 0, \\ r(r-1)x^r - 4rx^r + 4x^r &= 0, \\ (r^2 - 5r + 4)x^r &= 0. \end{aligned}$$

But $x > 0$ and so $x^r \neq 0$ and $(r^2 - 5r + 4) = 0$. This quadratic equation has two distinct real roots $r_1 = 4$ and $r_2 = 1$. Thus $y_1(x) = x^4$ and $y_2(x) = x$ are two solutions to the DE.

Existence and Uniqueness are very important concepts in DEs theory. Existence of a solution to the DE is supported by theorems stating that under certain restrictions the equation has at least one solution. Uniqueness discusses the conditions must be satisfied so that the problem has only one solution.

It's important to mention that the methods for solving DEs do not always give an explicit solution for the equation, Here we accept implicit solutions $\phi(x,y)=0$.

Example 1.3 Show that the relation

$$y-\ln y = x^2+1$$

implicitly defines a solution to the DE

$$y' = \frac{2xy}{y-1}.$$

Solution: Using the technique of implicit differentiation we obtain

$$\begin{aligned}
\frac{d}{dx}\left(y-\ln y\right) &= \frac{d}{dx}\left(x^2+1\right), \\
y' - \frac{1}{y}y' &= 2x, \\
\left(1-\frac{1}{y}\right)y' &= 2x, \\
\left(\frac{y-1}{y}\right)y' &= 2x.
\end{aligned}$$

This gives that $y' = \frac{2xy}{y-1}$ which is identical to the DE. Thus, the relation $y-\ln y = x^2+1$ is an implicit solution on some interval to the DE.

Definition 1.2 By an initial value problem for the nth-order DE

$$y^{(n)} = F(x,y,y',\ldots,y^{(n-1)}),$$

we mean: Find a solution to the DE on an interval I that satisfies at x_0 the n initial conditions (ICs):

$$\begin{aligned}
y(x_0) &= \alpha_0, \\
y'(x_0) &= \alpha_1, \\
&\vdots \\
y^{(n-1)}(x_0) &= \alpha_{n-1},
\end{aligned}$$

where $x_0 \in I$ and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are given constants. Initial value problems are denoted by IVPs.

For example, the following are IVPs:

$$\left.\begin{array}{ll} y' + 3y = x^2, & \longrightarrow \ DE \\ y(1) = 4. & \longrightarrow \ IC \end{array}\right\} \longrightarrow IVP$$

$$\left.\begin{array}{ll} y'' + 4y' + y^2 = 0, & \longrightarrow \ DE \\ y(0) = 4, \quad y'(0) = -2. & \longrightarrow \ ICs \end{array}\right\} \longrightarrow IVP$$

$$\left.\begin{array}{ll} y''' + y'' - 3y = 3x, & \longrightarrow \ DE \\ y(-1) = 0, \quad y'(-1) = 1, \ y''(-1) = 3. & \longrightarrow \ ICs \end{array}\right\} \longrightarrow IVP$$

Example 1.4 Show that $\phi(x) = e^x$ is a solution to the IVP

$$y' - y = 0,$$
$$y(0) = 1.$$

Solution: Notice that $\phi(x) = e^x$ and $\phi'(x) = e^x$ are all defined on $(-\infty, \infty)$. Substituting in the DE gives

$$y' - y = e^x - e^x = 0,$$

which holds for all $x \in (-\infty, \infty)$. Thus $\phi(x) = e^x$ is a solution to the DE and $\phi(0) = e^0 = 1$ which satisfies the IC. So, $\phi(x) = e^x$ is a solution to the IVP.

Example 1.5 Given that $y(x) = C_1 e^x + C_2 e^{-2x}$ is a solution to the DE

$$y'' + y' - 2y = 0,$$

for any choice of the constants C_1 and C_2. Determine C_1 and C_2 so that the initial conditions are satisfied

$$y(0) = 2, \qquad y'(0) = 1.$$

Solution: To determine the constants C_1 and C_2, we first compute $y'(x)$ to get

$$y'(x) = C_1 e^x - 2C_2 e^{-2x}.$$

Substituting in our initial conditions gives the following system of equations:

$$y(0) = 2 \Longrightarrow C_1 + C_2 = 2,$$
$$y'(0) = 1 \Longrightarrow C_1 - 2C_2 = 1.$$

Subtracting the last two equations yields $3C_2 = 1$, so $C_2 = 1/3$. Since $C_1 + C_2 = 2$, we find $C_1 = 5/3$. Hence, the solution to the initial value problem is

$$y(x) = \frac{5}{3}e^x + \frac{1}{3}e^{-2x}.$$

Chapter 2

First-Order Differential Equations

The general form for a first-order DE is

$$y' = f(x, y),$$

where x is an independent variable and $y = y(x)$ is the dependent variable.

Many physical problems, when formulated mathematically, lead to first-order differential equations or initial value problems. Several of these are discussed in Chapter 3. In this chapter we learn how to recognize and obtain solutions for some special types of first-order equations. We begin by studying linear equations, then separable equations, and then exact equations. The methods for solving these are the most basic. In some sections, we illustrate how devices such as integrating factors, substitutions, and transformations can be used to transform certain equations into either separable, exact, or linear equations that we can solve. Through our discussion of these special types of equations, you will gain insight into the behavior of solutions to more general equations and the possible difficulties in finding these solutions.

2.1 Existence and Uniqueness

Whether we're looking for exact solutions or numerical approximations, it's useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations.

Some terminology: an open rectangle R is a set of points (x, y) such that

$$R = \{(x, y) : a < x < b,\ c < y < d\}$$

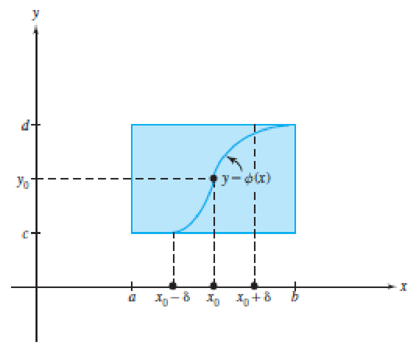


Fig. 2.1

"Open" means that the boundary rectangle isn't included in R.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first-order nonlinear differential equations.

Theorem 2.1

(a) If f is continuous on an open rectangle

$$R = \{(x, y) : a < x < b,\ c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has at least one solution $\phi(x)$ in some interval $I = (x_0 - \delta, x_0 - \delta)$, where δ is a positive number.

(b) If both f and $\frac{\partial f}{\partial y}$ are continuous on R then the IVP has a unique solution $\phi(x)$ in some interval $I = (x_0 - \delta, x_0 - \delta)$, where δ is a positive number.

It's important to understand exactly what Theorem 2.1 says:

- Part (a) is an existence theorem. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that the IVP may have.

- Part (b) is a uniqueness theorem. It guarantees that the IVP has a unique solution on some open interval I that contains x_0. However, if $I \neq (-\infty, \infty)$, The IVP may have more than one solution on a larger interval that contains I.

Example 2.1 Consider the IVP

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \qquad y(1) = 2.$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{2y\left(1 + 2x^2\right)}{\left(1 + x^2 + y^2\right)^2}$$

are continuous for all (x, y), Theorem 2.1 implies that the IVP has a unique solution on some open interval that contains $x_0 = 1$.

Example 2.2 Consider the initial value problem

$$y' = y^{1/5}, \qquad y(0) = 0.$$

The function $f(x, y) = y^{1/5}$ is continuous everywhere, but $\frac{\partial f}{\partial y} = \frac{1}{5y^{4/5}}$ is not continuous when $y = 0$ (i.e., points on the x-axis). The continuity of f does assure the existence of solutions. On the other hand, any rectangle containing $(0, 0)$ must contain points on the x-axis, so Part (b) of Theorem 2.1 does not apply to this problem and no conclusion can be drawn from it. Note that the IVP has at least two solutions, namely,

$$y_1(x) = \left(\frac{4}{5}(x + C)\right)^{5/4},$$
$$y_2(x) = -\left(\frac{4}{5}(x + C)\right)^{5/4}.$$

Example 2.3 Consider the IVP

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}, \qquad y(0) = -1.$$

Here

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y - 1)} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{3x^2 + 4x + 2}{2(y - 1)^2}.$$

Thus both these functions are continuous everywhere except on the line $y = 1$. Consequently, a rectangle can be drawn about the initial point $(0, -1)$

in which both f and $\frac{\partial f}{\partial y}$ are continuous. Therefore Theorem 2.1 guarantees that the IVP has a unique solution in some interval about $x_0 = 0$. However, even though the rectangle can be stretched infinitely far in both the positive and negative x directions, this does not necessarily mean that the solution exists for all x. Indeed, the IVP has the solution

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

To determine the interval in which the solution is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$.

Now suppose we change the initial condition to $y(0) = 1$. The initial point now lies on the line $y = 1$ so no rectangle can be drawn about it within which f and $\frac{\partial f}{\partial y}$ are continuous. Consequently, Theorem 2.1 says nothing about possible solutions of this modified problem. However, the functions

$$y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x}$$

are solutions as they satisfy the given differential equation for $x > 0$ and also satisfy the initial condition $y(0) = 1$.

2.2 Linear Equations

A linear first-order equation is an equation that can be expressed in the form

$$a_1(x)y' + a_2(x)y = a_3(x), \tag{2.1}$$

where $a_1(x)$, $a_2(x)$, and $a_3(x)$ depend only on the independent variable x, not on y. In other words, a first-order DE is linear if y and y' appear in additive combinations of their first powers.

For example, the DE

$$x^2y' + (\sin x)\, y = \frac{1}{1 + x^2}$$

is linear with $a_1(x) = x^2$, $a_2(x) = \sin x$, and $a_3(x) = \frac{1}{1+x^2}$.

However, the DE

$$y' - xy^3 = e^{-3x}$$

is not linear due to the presence of the y^3 term.

If we divide both sides of the DE (2.1) by $a_1(x)$, then we can write the general first-order linear equation in the form

$$y' + p(x)y = g(x), \tag{2.2}$$

where $p(x) = a_2(x)/a_1(x)$ and $g(x) = a_3(x)/a_1(x)$.

Let $\mu(x)$ be a function to be determined later. Multiplying Eq. (2.2) by $\mu(x)$, we obtain

$$\mu(x)y' + p(x)\mu(x)y = \mu(x)g(x).$$

The last equation amounts to the same as

$$\mu(x)y' + \underbrace{\mu'(x)y - \mu'(x)y}_{0} + p(x)\mu(x)y = \mu(x)g(x),$$

or

$$(\mu(x)y' + \mu'(x)y) + p(x)\mu(x)y - \mu'(x)y = \mu(x)g(x).$$

Then we have

$$(\mu(x)y)' + (p(x)\mu(x) - \mu'(x))y = \mu(x)g(x). \tag{2.3}$$

We will determine $\mu(x)$ such that

$$p(x)\mu(x) - \mu'(x) = 0.$$

Then we have

$$\frac{\mu'(x)}{\mu(x)} = p(x),$$

and as a result of integrating both sides with respect to x, we obtain that

$$\ln \mu(x) = \int p(x)dx + k.$$

By choosing the arbitrary constant k to be zero, we obtain the simplest possible function for $\mu(x)$, namely,

$$\mu(x) = e^{\int p(x)dx}.$$

Note that $\mu(x)$ is positive for all x. Returning to Eq. (2.3), we have

$$(\mu(x)y)' = \mu(x)g(x).$$

Hence

$$\mu(x)y = \int \mu(x)g(x)dx + C,$$

so the general solution of Eq. (2.2) is

$$y(x) = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}.$$

The function $\mu(x)$ is called an integrating factor to the DE (2.2).

We can summarize the method for solving linear equations as follows.

- Make sure the equation is written in standard form
$$y' + p(x)y = g(x),$$
and then identify $p(x)$ and $g(x)$.
- Find the integrating factor $\mu(x)$ by the formula
$$\mu(x) = e^{\int p(x)dx}.$$
- The general solution can be obtained using the formula
$$y(x) = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}.$$

Example 2.4 Find the general solution to the DE

$$xy' - 2y = x^2.$$

Solution: To put this linear equation in standard form, we divide by x to obtain

$$y' - \frac{2}{x}y = x.$$

Here $p(x) = -\frac{2}{x}$ and $g(x) = x$. Thus, an integrating factor is

$$\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}.$$

We now find the general solution

$$y(x) = \frac{\int \mu(x)g(x)dx + C}{\mu(x)} = \frac{\int x^{-2}xdx + C}{x^{-2}} = x^2\left(\ln|x| + C\right).$$

Several solution curves (for $C = -2, -1, 0, 1, 2, 3,$ and 4) are shown in Fig. 2.2.

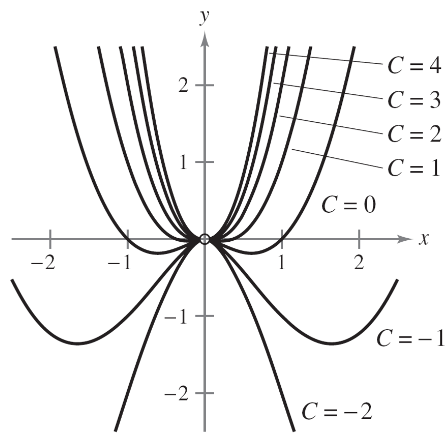


Fig. 2.2

Example 2.5 Solve the DE

$$\frac{dy}{dt} - 2y = 4 - t.$$

Solution: The DE is in standard form. Here $p(t) = -2$ and $g(t) = 4 - t$. The integrating factor for the DE is

$$\mu(t) = e^{\int p(t)dt} = e^{\int -2dt} = e^{-2t}.$$

Thus the general solution of the DE is

$$\begin{aligned} y(t) &= \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\int \left(4e^{-2t} - te^{-2t}\right)dt + C}{e^{-2t}} \\ &= \frac{-2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + C}{e^{-2t}} \\ &= -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}. \end{aligned}$$

Example 2.6 Solve the initial value problem

$$xy' + 2y = 4x^2,$$
$$y(1) = 2.$$

Solution: Rewriting the DE in the standard form, we have

$$y' + \frac{2}{x}y = 4x,$$

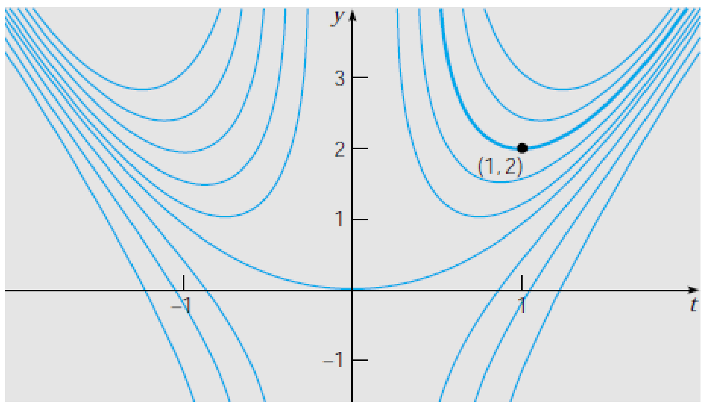


Fig. 2.3

so $p(x) = \frac{2}{x}$ and $g(x) = 4x$. In order to solve the DE, we first compute the integrating factor $\mu(x)$:

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln|x|} = x^2.$$

It follows that

$$y(x) = \frac{\int x^2(4x)dt + C}{x^2} = x^2 + Cx^{-2},$$

is the general solution of the DE. Integral curves of the DE for several values of C are shown in Fig. 2.3. To satisfy the initial condition

$$y(1) = 2 \Longrightarrow 1 + C = 2,$$

it is necessary to choose $C = 1$; thus

$$y(x) = x^2 + x^{-2}$$

is the solution of the initial value problem.

2.3 Bernoulli Equations

A first order DE that can be written in the form

$$y' + p(x)y = g(x)y^n, \qquad n \neq 0, 1, \tag{2.4}$$

where $p(x)$ and $g(x)$ are continuous on an open interval I, is called Bernoulli equation. Note that (2.4) is not linear.

Example 2.7

(a) The DE $y' + 4y = \frac{x}{y^2}$ can be rewritten as $y' + 4y = xy^{-2}$. This is a Bernoulli equation with $n = -2$.
(b) The DE $y' - \frac{2}{x}y = \sqrt{y}$ can be rewritten as $y' - \frac{2}{x}yy = y^{1/2}$ and so it is a Bernoulli equation with $n = 1/2$.

To solve Bernoulli equation, we use the substitution

$$v = y^{1-n}.$$

Differentiate v to get

$$v' = (1-n)y^{-n}y'.$$

Multiply Bernoulli equation (2.4) by $(1-n)y^{-n}$ to have

$$(1-n)y^{-n}y' + (1-n)p(x)y^{1-n} = (1-n)g(x).$$

The last equation becomes

$$v' + (1-n)p(x)v = (1-n)g(x).$$

This is a linear DE on v. Solve it to find v. Then using that $v = y^{1-n}$ we get an implicit solution to Bernoulli equation (2.4).

Example 2.8 Solve the DE

$$y' - \frac{2}{x}y = \frac{x^2}{y^2}. \tag{2.5}$$

Solution: The DE can be written as

$$y' - \frac{2}{x}y = x^2y^{-2}.$$

This is a Bernoulli equation with $n = -2$. To transform (2.5) into a linear equation, we make the substitution

$$v = y^{1-n} = y^3.$$

Differentiate to obtain

$$v' = 3y^2y'.$$

Multiply the DE (2.5) by $3y^2$ to get

$$3y^2y' - \frac{6}{x}y^3 = 3x^2,$$
$$v' - \frac{6}{x}v = 3x^2. \tag{2.6}$$

Equation (2.6) is linear with $p(x) = -\frac{6}{x}$ and $g(x) = 3x^2$. So we can solve it for v using the method discussed in Section 2.2. When we do this, it turns out that the integrating factor is

$$\mu(x) = e^{\int -\frac{6}{x}\,dx} = e^{-6\ln|x|} = x^{-6}, \qquad x > 0,$$

and

$$v = \frac{\int x^{-6}\left(3x^2\right)dx + C}{x^{-6}} = -x^3 + Cx^6.$$

Substituting $v = y^3$ gives the solution

$$y^3 = -x^3 + Cx^6.$$

In this case we can obtain an explicit solution

$$y = \left(-x^3 + Cx^6\right)^{1/3}, \qquad x > 0.$$

Example 2.9 Solve the IVP

$$y' - \frac{3}{x}y = x^2\sqrt{y}, \tag{2.7}$$
$$y(1) = 2.$$

Solution: This is a Bernoulli equation with $n = 1/2$. We use the substitution

$$v = y^{1-n} = y^{1/2}$$

and so

$$v' = \frac{1}{2}y^{-1/2}y'.$$

When we multiply the DE (2.7) by $\frac{1}{2}y^{-1/2}$ we obtain the linear equation

$$v' - \frac{3}{2x}v = \frac{1}{2}x^2,$$

with $p(x) = -\frac{3}{2x}$ and $g(x) = \frac{1}{2}x^2$. To solve this linear equation we first find

$$\mu(x) = e^{\int -\frac{3}{2x}\,dx} = e^{-\frac{3}{2}\ln|x|} = x^{-3/2}, \qquad x > 0,$$

then the general solution is given by

$$v = \frac{\int x^{-3/2}\left(\frac{1}{2}x^2\right)dx + C}{x^{-3/2}} = \frac{1}{3}x^3 + Cx^{3/2}.$$

Using that $v = y^{1/2}$, we obtain an implicit solution to (2.7),

$$y^{1/2} = \frac{1}{3}x^3 + Cx^{3/2}.$$

Inserting the given IC $y(1) = 2$ yields

$$\sqrt{2} = \frac{1}{3} + C,$$
$$C = \sqrt{2} - \frac{1}{3}.$$

Thus the solution of the IVP is given implicitly by

$$y^{1/2} = \frac{1}{3}x^3 + \left(\sqrt{2} - \frac{1}{3}\right)x^{3/2}.$$

2.4 Separable Equations

A first order DE that can be rewritten to separate the variables x and y on opposite sides of the equation, as in

$$h(y)dy = g(x)dx, \tag{2.8}$$

is called a separable equation.

Example 2.10 Determine whether the given differential equation is separable

$$\frac{dy}{dx} = x\ln(y^{2x}) + 8x^2.$$

Solution: The differential equation can be written as

$$\begin{aligned}\frac{dy}{dx} &= x(2x)\ln(y) + 8x^2 \\ \frac{dy}{dx} &= 2x^2\ln(y) + 8x^2 \\ \frac{dy}{dx} &= 2x^2\left(\ln(y) + 4\right) \\ \frac{1}{\ln(y)+4}dy &= 2x^2 dx,\end{aligned}$$

then it has the form (2.8) and is therefore separable.

Letting $H(y)$ and $G(x)$ denote antiderivatives (indefinite integrals) of $h(y)$ and $g(x)$, respectively,

$$H'(y) = h(y), \qquad G'(x) = g(x).$$

Now Eq. (2.8) can be written as

$$H'(y)\frac{dy}{dx} = G'(x).$$

By the chain rule for differentiation, the left-hand side is the derivative of the composite function $H(y(x))$:

$$\frac{d}{dx}H(y(x)) = H'(y(x))\frac{dy}{dx}.$$

Thus,

$$\frac{d}{dx}H(y(x)) = G'(x)$$

and consequently $H(y(x))$ and $G(x)$ are two functions of x that have the same derivative. Therefore, they differ by a constant:

$$H(y(x)) = G(x) + C.$$

This can be used to construct implicit solutions.

In summary, all that is needed to solve a separable equation is to integrate the left side with respect to y and the right side with respect to x. Once this done, we get an implicit solution.

Example 2.11 Solve the DE

$$y' = \frac{x^2+1}{y^3+2}.$$

Solution: We separate the variables and rewrite the DE in the form

$$\frac{dy}{dx} = \frac{x^2+1}{y^3+2},$$
$$\left(y^3+2\right) dy = \left(x^2+1\right) dx.$$

Thus the DE is separable. In order to solve it, we integrate both sides to have

$$\int \left(y^3+2\right) dy = \int \left(x^2+1\right) dx,$$
$$\frac{1}{4}y^4 + 2y = \frac{1}{3}x^3 + x + C.$$

This is considered an implicit solution to the DE.

Example 2.12 Solve the DE

$$y' = \frac{x^2y-4y}{x+2}.$$

Solution: This problem will require a little work to get it separated, so let's do that first,

$$\frac{dy}{dx} = \frac{\left(x^2-4\right)y}{x+2},$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{(x-2)(x+2)}{x+2},$$
$$\frac{1}{y}dy = (x-2)\,dx.$$

It's a separable DE. Integrate the left side with respect to y and the right side with respect to x to get

$$\ln|y| = \frac{x^2}{2} - 2x + C_1.$$

Exponentiating the last equation, we have

$$|y| = e^{\frac{x^2}{2}-2x+C_1} = e^{C_1}e^{\frac{x^2}{2}-2x} = Ce^{\frac{x^2}{2}-2x}$$

Now, depending on the values of y, we have

$$y = \pm Ce^{\frac{x^2}{2}-2x},$$

where the choice of sign depends on the values of x and y. Because C is a positive constant (recall that $C = e^{C_1} > 0$), we can replace $\pm C$ by K, where K now represents an arbitrary nonzero constant. We then obtain

$$y = Ke^{\frac{x^2}{2} - 2x}.$$

Example 2.13 Find the solution to the IVP

$$\frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2}, \qquad y(0) = 1.$$

Solution: Observe that $y = 0$ is a solution of the given differential equation. To find other solutions, assume that $y \neq 0$ and write the differential equation in the form

$$\frac{1 + 2y^2}{y} dy = \cos x dx.$$

Then, integrating the left side with respect to y and the right side with respect to x, we obtain

$$\int \left(\frac{1}{y} + 2y \right) dy = \int \cos x dx$$
$$\ln |y| + y^2 = \sin x + C.$$

To satisfy the initial condition we substitute $x = 0$ and $y = 1$ in the last equation; this gives $C = 1$. Hence the solution of the initial value problem is given implicitly by

$$\ln |y| + y^2 = \sin x + 1.$$

Some integral curves of the given differential equation, including the solution of the initial value problem, are shown in Fig. 2.4.

2.5 Homogeneous Equations

If the right-hand side of the equation

$$y' = f(x, y)$$

can be expressed as a function of the ratio y/x alone, $G\left(\frac{y}{x}\right)$, then we say that the equation is homogeneous.

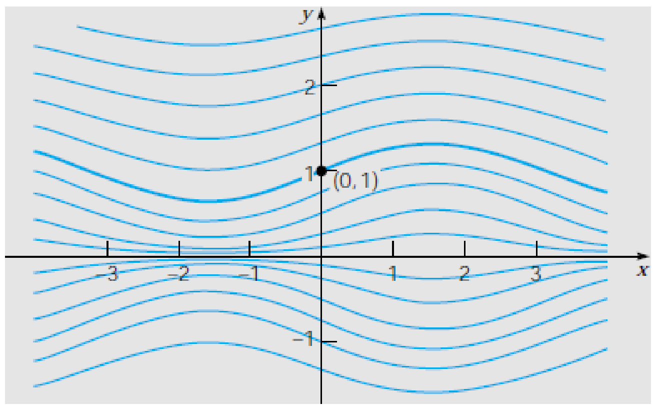


Fig. 2.4

For example, the DE

$$y' = \frac{y - 4x}{x - 3y} \tag{2.9}$$

can be written in the form

$$y' = \frac{\left(\frac{y}{x}\right) - 4}{1 - 3\left(\frac{y}{x}\right)} = G\left(\frac{y}{x}\right),$$

where $G(v) = \frac{v-4}{1-3v}$. Since we have expressed the right-hand side of the equation as a function of the ratio y/x, then equation (2.9) is homogeneous.

Homogeneous equations can always be transformed into separable equations by a change of the dependent variable. To do so, we make the substitution

$$v = \frac{y}{x}.$$

Our homogeneous equation now has the form

$$\frac{dy}{dx} = G\left(\frac{y}{x}\right). \tag{2.10}$$

Keeping in mind that both v and y are functions of x, we use the product rule for differentiation to deduce from $y = vx$ that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We then substitute the above expression for dy/dx into equation (2.10) to obtain

$$\begin{aligned} v + x\frac{dv}{dx} &= G\left(v\right), \\ \frac{1}{G\left(v\right) - v}dv &= \frac{1}{x}dx. \end{aligned} \tag{2.11}$$

The new equation (2.11) is separable, and we can obtain its implicit solution from

$$\int \frac{1}{G\left(v\right) - v}dv = \int \frac{1}{x}dx$$

All that remains to do is to express the solution in terms of the original variables x and y.

Example 2.14 Solve the DE

$$\left(3x^2 - y^2\right)dx + \left(xy - x^3y^{-1}\right)dy = 0 \tag{2.12}$$

Solution: If we express (2.12) in the derivative form

$$\frac{dy}{dx} = \frac{y^2 - 3x^2}{xy - x^3y^{-1}} = \frac{\left(\frac{y}{x}\right)^2 - 3}{\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^{-1}} = G\left(\frac{y}{x}\right), \tag{2.13}$$

then we see that the right-hand side of (2.13) is a function of just y/x. Thus, equation (2.12) is homogeneous. Now let $v = \frac{y}{x}$ and recall that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. With these substitutions, equation (2.13) becomes

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{v^2 - 3}{v - v^{-1}}, \\ \frac{1 - v^2}{2v}dv &= \frac{1}{x}dx \end{aligned}$$

The above equation is separable, and, on separating the variables and integrating, we obtain

$$\begin{aligned} \int \frac{1 - v^2}{2v}dv &= \int \frac{1}{x}dx, \\ \frac{1}{2}\left(\ln|v| - \frac{v^2}{2}\right) &= \ln|x| + C. \end{aligned}$$

Finally, we substitute $v = y/x$ to get

$$\frac{1}{2}\left(\ln\left|\frac{y}{x}\right| - \frac{\left(\frac{y}{x}\right)^2}{2}\right) = \ln|x| + C$$

as an implicit solution to equation (2.12).

2.6 Exact Equations

Any first-order DE $y' = f(x, y)$ can be rewritten in the form

$$M(x, y)dx + N(x, y)dy = 0. \tag{2.14}$$

For example, the DE

$$y' = \frac{2x^2y + 3x - 1}{xy^3 + e^y}$$

can be rewritten in the form

$$\left(1 - 2x^2y - 3x\right) dx + \left(xy^3 + e^y\right) dy = 0$$

Here $M(x, y) = \left(1 - 2x^2y - 3x\right)$ and $N(x, y) = \left(xy^3 + e^y\right)$.

Example 2.15 Consider the DE

$$(2x + y^2)dx + (2xy)dy = 0. \tag{2.15}$$

Observe that

$$(2x + y^2)dx + (2xy)dy = \frac{\partial}{\partial x}(x^2 + xy^2)dx + \frac{\partial}{\partial y}(x^2 + xy^2)dy.$$

Thus the left-hand side of Eq. (2.15) is identified as a total differential,

$$(2x + y^2)dx + (2xy)dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF(x, y)$$

where $F(x, y) = (x^2 + xy^2)$. In order to solve the DE (2.15), we need to solve

$$dF = 0.$$

Thus, the solutions of the DE are given (implicitly) by the level curves

$$F(x, y) = C$$

for an arbitrary constant C.

We'll say that the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is exact on an open rectangle R if there's a function $F(x,y)$ such that F_x and F_y are continuous, and

$$F_x(x,y) = M(x,y) \qquad \text{and} \qquad F_y(x,y) = N(x,y) \tag{2.16}$$

for all (x,y) in R. This usage of "exact" is related to its usage in calculus, where the expression

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

is the exact (total) differential of F, denoted by dF.

Example 2.15 shows that it's easy to solve (2.15) if it's exact and we know a function F that satisfies (2.16). The important questions are:

Q1. Given an equation (2.14), how can we determine whether it's exact?
Q2. If (2.14) is exact, how do we find a function F satisfying (2.16)?

A systematic way of determining whether a given differential equation is exact is provided by the following theorem.

Theorem 2.2 *Let the functions N, M, M_y, N_x be continuous in the rectangular region $R = \{(x,y) : a < x < b, c < y < d\}$. Then the DE*

$$M(x,y)dx + N(x,y)dy = 0$$

is exact in R if and only if the condition

$$M_y(x,y) = N_x(x,y)$$

holds for all (x,y) in R.

Example 2.16 Determine whether the equation is exact

(a) $\left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0.$
Here $M(x,y) = ye^{xy} - \frac{1}{y}$ and $N(x,y) = xe^{xy} + \frac{x}{y^2}$. It is easy to find that

$$M_y(x,y) = xye^{xy} + e^{xy} + \frac{1}{y^2} = N_x(x,y),$$

so the given equation is exact.

(b) $\left(3xy+y^2\right)dx+\left(x^2+xy\right)dy=0$.
Here $M(x,y)=3xy+y^2$ and $N(x,y)=x^2+xy$. It is easy to find that

$$M_y(x,y)=3x+2y, \qquad N_x(x,y)=2x+y.$$

Since $M_y \neq N_x$, the given equation is not exact.

The next example illustrates a possible method for finding a function $F(x,y)$ that satisfies the condition $F_x(x,y)=M(x,y)$ and $F_y(x,y)=N(x,y)$ if $M(x,y)dx+N(x,y)dy=0$ is exact.

Example 2.17 Solve

$$\left(4x^3y^3+3x^2\right)dx+\left(3x^4y^2+6y^2\right)dy=0. \tag{2.17}$$

Solution: Here $M(x,y)=4x^3y^3+3x^2$ and $N(x,y)=3x^4y^2+6y^2$. It is easy to find that

$$M_y(x,y)=12x^3y^2=N_x(x,y),$$

for all (x,y). Therefore Theorem 2.2 implies that the given equation is exact. Consequently there's a function F such that

$$F_x(x,y)=M(x,y)=4x^3y^3+3x^2 \tag{2.18}$$

and

$$F_y(x,y)=N(x,y)=3x^4y^2+6y^2 \tag{2.19}$$

for all (x,y). To find F, we integrate (2.18) with respect to x to obtain

$$F(x,y)=\int\left(4x^3y^3+3x^2\right)dx=x^4y^3+x^3+\phi(y), \tag{2.20}$$

where $\phi(y)$ is the "constant" of integration ($\phi(y)$ independent of x, the variable of integration.) To determine ϕ so that F also satisfies (2.19), assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$3x^4y^2+\phi'(y)=3x^4y^2+6y^2$$

and so

$$\phi'(y)=6y^2.$$

Integrate this with respect to y and take the constant of integration to be zero because we're interested only in finding one F that satisfies (2.18) and (2.19). This yields

$$\phi(y) = 2y^3.$$

Hence, from (2.20), we have $F(x,y) = x^4y^3 + x^3 + 2y^3$, and the solution to equation (2.17) is given implicitly by $F(x,y) = C$;

$$x^4y^3 + x^3 + 2y^3 = C.$$

We can also obtain the function F in a different way. Instead of first integrating (2.18) with respect to x, we could begin by integrating (2.19) with respect to y to obtain

$$F(x,y) = \int \left(3x^4y^2 + 6y^2\right) dy = x^4y^3 + 2y^3 + \psi(x),$$

where ψ is an arbitrary function of x. To determine ψ, we assume that ψ is differentiable and differentiate F with respect to x, which yields

$$F_x(x,y) = 4x^3y^3 + \psi'(x).$$

Comparing this with (2.18) shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

Hence $F(x,y) = x^4y^3 + 2y^3 + x^3$.

Many equations can be conveniently solved by either of the two methods used in the last example. However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

Example 2.18 Solve the equation

$$\left(ye^{xy}\tan x + e^{xy}\sec^2 x\right) dx + \left(xe^{xy}\tan x\right) dy = 0. \tag{2.21}$$

Solution: Here

$$M(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x \qquad \text{and} \qquad N(x,y) = xe^{xy}\tan x.$$

It is easy to see that

$$M_y = xye^{xy}\tan x + e^{xy}\tan x + xe^{xy}\sec^2 x = N_x,$$

so the given equation is exact. Thus there is a function $F(x, y)$ such that

$$F_x(x, y) = M(x, y) = ye^{xy}\tan x + e^{xy}\sec^2 x \tag{2.22}$$

and

$$F_y(x, y) = N(x, y) = xe^{xy}\tan x. \tag{2.23}$$

It's difficult to integrate (2.22) with respect to x, but easy to integrate (2.23) with respect to y. This yields

$$F(x, y) = \int xe^{xy}\tan x dy = e^{xy}\tan x + \phi(x). \tag{2.24}$$

Substituting in (2.22) gives

$$ye^{xy}\tan x + e^{xy}\sec^2 x + \phi'(x) = ye^{xy}\tan x + e^{xy}\sec^2 x.$$

Thus $\phi'(x) = 0$ and $\phi(x)$ is a constant, which we can take to be zero. Substituting for $\phi(x)$ in Eq. (2.24) gives

$$F(x, y) = e^{xy}\tan x.$$

Hence solutions of Eq. (2.21) are given implicitly by

$$e^{xy}\tan x = C_1.$$

In this case we can solve explicitly for y to obtain

$$y = \frac{C + \ln|\cot x|}{x}.$$

2.7 Special Integrating Factors

It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. Recall that this is the procedure that we used in solving linear equations in Section 2.2. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{2.25}$$

by a function $\mu(x, y)$ and then try to choose μ so that the resulting equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \tag{2.26}$$

is exact. Such a function $\mu(x, y)$ is called an integrating factor for (2.25). By Theorem 2.2 Eq. (2.26) is exact if and only if

$$(\mu M)_y = (\mu N)_x\,. \tag{2.27}$$

Keeping in mind that M and N are given functions, Eq. (2.27) states that the integrating factor μ must satisfy the first order partial differential equation

$$\begin{aligned} M\mu_y + M_y\mu = N\mu_x + N_x\mu = 0,\\ M\mu_y - N\mu_x + (M_y - N_x)\,\mu = 0. \end{aligned} \tag{2.28}$$

If a function μ satisfying Eq. (2.28) can be found, then Eq. (2.26) will be exact. A partial differential equation of the form (2.28) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of Eq. (2.25).

Unfortunately, Eq. (2.28), which determines the integrating factor μ, is usually at least as difficult to solve as the original equation (2.25). Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both.

Let's assume that Eq. (2.25) has an integrating factor that depends only on x; that is, $\mu = \mu(x)$. In this case $\mu_y = 0$ and Eq. (2.28) reduces to the separable equation

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}. \tag{2.29}$$

Then we can solve the separable equation (2.29) to obtain

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}.$$

In a similar fashion, if Eq. (2.25) has an integrating factor that depends only on y, then $\mu_x = 0$ and Eq. (2.28) reduces to the separable equation

$$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}. \tag{2.30}$$

Then we can solve the separable equation (2.30) to obtain

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

We summarize these observations in the following procedure:

Method for Finding Special Integrating Factors:

If the DE $M(x, y)dx + N(x, y)dy = 0$ is not exact, we consider

- If $\frac{M_y - N_x}{N}$ is a function of just x, then an integrating factor is given by

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}. \tag{2.31}$$

- If $\frac{N_x - M_y}{M}$ is a function of just y, then an integrating factor is given by

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}. \tag{2.32}$$

Example 2.19 Find an integrating factor for the equation

$$\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right) dx + \left(3x^2y^2 + 4y\right) dy = 0 \tag{2.33}$$

and solve the equation.

Solution: Here

$$M(x, y) = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x, \qquad N(x, y) = 3x^2y^2 + 4y,$$

and

$$M_y = 6xy^2 - 6x^3y^2 - 8xy, \qquad N_x = 6xy^2.$$

Since $M_y \neq N_x$, so Eq. (2.33) is not exact. However,

$$\frac{M_y - N_x}{N} = \frac{-6x^3y^2 - 8xy}{3x^2y^2 + 4y} = \frac{-2x\left(3x^2y^2 + 4y\right)}{3x^2y^2 + 4y} = -2x$$

is a function of only x. So an integrating factor for (2.33) is given by formula (2.31). That is,

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int -2x dx} = e^{-x^2}.$$

When we multiply (2.33) by $\mu(x) = e^{-x^2}$, we get the exact equation

$$e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right) dx + e^{-x^2}\left(3x^2y^2 + 4y\right) dy = 0.$$

Thus there is a function $F(x, y)$ such that

$$F_x(x, y) = e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right) \tag{2.34}$$

and

$$F_y(x,y) = e^{-x^2}\left(3x^2y^2 + 4y\right). \tag{2.35}$$

Integrating (2.35), we obtain

$$F(x,y) = \int e^{-x^2}\left(3x^2y^2 + 4y\right)dy = e^{-x^2}\left(x^2y^3 + 2y^2\right) + \phi(x).$$

Now we can use (2.34) to get

$$\begin{aligned} F_x(x,y) &= e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right), \\ e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2\right) + \phi'(x) &= e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right), \end{aligned}$$

and so

$$\phi'(x) = 2xe^{-x^2}.$$

A direct integration by substitution $\left(u = -x^2\right)$ yields

$$\phi(x) = \int 2xe^{-x^2}dx = -e^{-x^2}.$$

Hence solutions of Eq. (2.33) are given implicitly by

$$\begin{aligned} F(x,y) &= C, \\ e^{-x^2}\left(x^2y^3 + 2y^2\right) - e^{-x^2} &= C, \\ e^{-x^2}\left(x^2y^3 + 2y^2 - 1\right) &= C. \end{aligned}$$

Example 2.20 Solve the differential equation

$$2xy^3dx + \left(3x^2y^2 + x^2y^3 + 1\right)dy = 0. \tag{2.36}$$

Here,

$$M(x,y) = 2xy^3, \qquad N(x,y) = 3x^2y^2 + x^2y^3 + 1,$$

and

$$M_y(x,y) = 6xy^2, \qquad N_x(x,y) = 6xy^2 + 2xy^3.$$

Since $M_y \neq N_x$, the given equation is not exact. Let us first determine whether it has an integrating factor that depends on x only. On computing the quantity

$$\frac{M_y - N_x}{N} = \frac{-2xy^3}{3x^2y^2 + x^2y^3 + 1}$$

we find that $\frac{M_y - N_x}{N}$ is not a function of x only. However, the quantity

$$\frac{N_x - M_y}{M} = \frac{2xy^3}{2xy^2} = 1$$

is independent of x, so we can consider it as a function of y only, $1 = y^0$. Thus there is an integrating factor μ that is a function of y only. To obtain μ, we use formula (2.32),

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int 1 dy} = e^y.$$

Multiplying (2.36) by $\mu(y) = e^y$ yields the exact equation

$$2xy^3 e^y dx + \left(3x^2y^2 + x^2y^3 + 1\right) e^y dy = 0$$

This ensures that there is a function $F(x, y)$ such that

$$F_x(x, y) = 2xy^3 e^y \tag{2.37}$$

and

$$F_y(x, y) = \left(3x^2y^2 + x^2y^3 + 1\right) e^y. \tag{2.38}$$

Integrating (2.37) with respect to x yields

$$F(x, y) = \int 2xy^3 e^y dx = x^2y^3 e^y + \phi(y).$$

Using (2.38) yields

$$3x^2y^2e^y + x^2y^3e^y + \phi'(y) = \left(3x^2y^2 + x^2y^3 + 1\right) e^y$$

and so $\phi'(y) = e^y$. Hence, $\phi(y) = e^y$ and consequently

$$F(x, y) = x^2y^3e^y + e^y = \left(x^2y^3 + 1\right) e^y.$$

We conclude that

$$\left(x^2y^3 + 1\right) e^y = C$$

is an implicit solution of (2.36).

Chapter 3

Mathematical Modeling

3.1 Orthogonal trajectories

Suppose that we have two families of curves given by

$$F(x, y, c) = 0 \qquad \text{and} \qquad G(x, y, k) = 0,$$

such that at any intersection of a curve of the family $F(x, y, c) = 0$ with a curve of the family $G(x, y, k) = 0$, the tangents of the curves are perpendicular. Then we say that the two families are orthogonal trajectories of each other. In this case, we have two families of curves that always intersect perpendicularly.

For example, the family of circles represented by $x^2 + y^2 = c$, with center at the origin, and the family $y = kx$ of straight lines through the origin, are orthogonal trajectories of each other, as shown in Fig. 3.1.

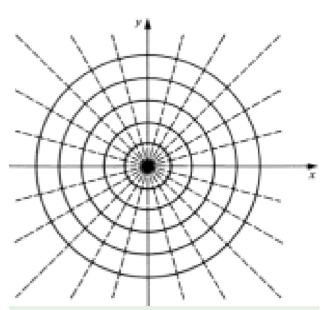


Fig. 3.1

Example 3.1 Find the orthogonal trajectories of the family

$$cx^2 - y^2 = 1. \tag{3.1}$$

Solution: To find the orthogonal trajectories, we follow the following four steps:

Step 1. Differentiate (3.1) implicitly with respect to x to obtain

$$2cx - 2y\frac{dy}{dx} = 0. \tag{3.2}$$

Step 2. Eliminate the parameter c. From (3.1) we have

$$c = \frac{1+y^2}{x^2}.$$

Substituting the value of c in (3.2) yields

$$\frac{dy}{dx} = \frac{1+y^2}{xy}. \tag{3.3}$$

This gives the differential equation of the family (3.1).

Step 3. Replace $\frac{dy}{dx}$ in (3.3) by $\frac{-1}{\frac{dy}{dx}}$. Orthogonal trajectories have the slope $\frac{-1}{\frac{dy}{dx}}$. Thus, to obtain the differential equation of the orthogonal trajectories, we replace $\frac{dy}{dx}$ in (3.3) by $\frac{-1}{\frac{dy}{dx}}$ to get

$$\frac{dy}{dx} = -\frac{xy}{1+y^2}. \tag{3.4}$$

Step 4. Solve the DE (3.4). It's easy to verify that Equation (3.4) is separable and so can be written as

$$\frac{1+y^2}{y}dy = -xdx.$$

Integrating the left-hand side with respect to y and the right-hand side with respect to x yields

$$\begin{aligned} \int \frac{1+y^2}{y}dy &= -\int xdx, \\ \int \left(\frac{1}{y} + y\right) dy &= -\frac{x^2}{2} + c_1, \\ \ln|y| + \frac{y^2}{2} &= -\frac{x^2}{2} + c_1, \\ 2\ln|y| + y^2 &= -x^2 + k. \end{aligned}$$

Thus, the required equation of orthogonal trajectories is given by $2\ln|y| + y^2 + x^2 = k$.

3.2 Newton's law of cooling

According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $T'(t)$ the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$T'(t) = -k\left(T(t) - T_m(t)\right). \tag{3.5}$$

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m. In either case, cooling or warming, it stands to reason that $k > 0$. This is due to the fact that the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the temperature decay constant of the medium.

This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it's reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it's a huge cauldron of molten metal.

To solve (3.5), we rewrite it as

$$T'(t) + kT(t) = -kT_m.$$

The solutions of this linear equation is

$$T(t) = T_m + Ce^{-kt}.$$

If $T(0) = T_0$, setting $t = 0$ here yields $C = T_0 - T_m$, so

$$T(t) = T_m + \left(T_0 - T_m\right)e^{-kt}. \tag{3.6}$$

Note that $T - T_m$ decays exponentially, with decay constant k.

Example 3.2 When a cake is removed from an oven, its temperature is measured at $300^\circ\ F$. Three minutes later its temperature is $200^\circ\ F$. How long will it take for the cake to cool off to a room temperature of 70° F? What is its temperature of the cake after 20 minutes?

Solution: In (3.5) we make the identification $T_m = 70$. We must then solve the initial-value problem

$$\begin{aligned} T' &= -k\,(T-70)\,, \\ T(0) &= 300. \end{aligned} \tag{3.7}$$

It's also given that $T(3) = 200$. Equation (3.7) is linear and has the solution given in (3.6)

$$\begin{aligned} T(t) &= T_m + (T_0 - T_m)\,e^{-kt}, \\ T(t) &= 70 + 230e^{-kt}. \end{aligned}$$

Finally, the measurement $T(3) = 200$ leads

$$\begin{aligned} 200 &= 70 + 230e^{-3k}, \\ e^{3k} &= \frac{230}{130}, \\ k &= \frac{1}{3}\ln\left(\frac{23}{13}\right) = 0.19018. \end{aligned}$$

Thus,

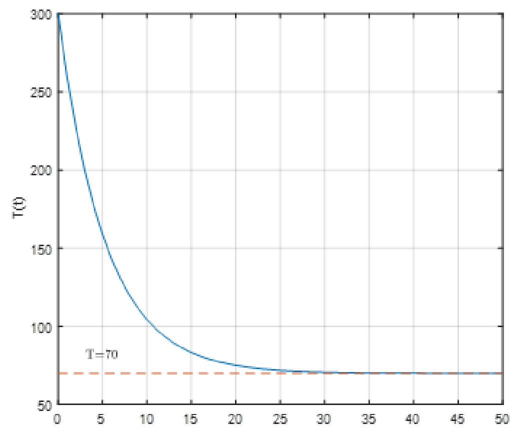
$$T(t) = 70 + 230e^{-0.19018t}. \tag{3.8}$$

Therefore the temperature of the cake after 20 minutes is

$$T(20) = 70 + 230e^{-0.19018(20)} \approx 75^\circ.$$

We note that (3.8) gives no finite solution to $T(t) = 70$, since $\lim_{t\to\infty} T(t) = 70$. The accompanying figure and table clearly show that the cake will be approximately at room temperature in about 45 minutes.

t (min)	*T(t)*
20	75.1267
25	71.9809
30	70.7654
35	70.2957
40	70.1142
45	70.0441
50	70.0170
55	70.0065



3.3 Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable t will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative dP/dt represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0 e^{rt}$. In this function, $P(t)$ represents the population at time t, P_0 represents the initial population (population at time $t = 0$), and the constant $r > 0$ is called the growth rate. One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r

usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the logistic differential equation is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{3.9}$$

The logistic equation was first published by Pierre Verhulst in 1845 . This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

The logistic differential equation is separable differential equation because it can be written as

$$\begin{aligned} \frac{dP}{dt} &= rP\left(\frac{K-P}{K}\right), \\ \frac{K}{P(K-P)}dP &= rdt. \end{aligned}$$

To solve this separable equations we integrate to have

$$\begin{aligned} \int \frac{K}{P(K-P)}dP &= \int rdt, \\ \int \left(\frac{1}{P} + \frac{1}{K-P}\right) dP &= rt + C, \\ \ln|P| - \ln|K-P| &= rt + C, \\ \ln\left|\frac{P}{K-P}\right| &= rt + C. \end{aligned}$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$\frac{P}{K-P} = e^{rt+C},$$

Solving the last equation for P gives

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}.$$

To determine the value of C_1, we use the IC: $P(0) = P_0$ in the last equation for $P(t)$ and solve for C_1:

$$C_1 = \frac{P_0}{K - P_0}.$$

This gives the formula for $P(t)$ as

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}} = \frac{P_0 K}{(K - P_0)\, e^{-rt} + P_0}. \tag{3.10}$$

Example 3.3 Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number P of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $P(4) = 50$.

Solution: Assuming that no one leaves the campus throughout the duration of the disease, we must solve the initial-value problem

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{1000}\right),$$
$$P(0) = 1.$$

We have immediately from (3.10) that

$$P(t) = \frac{1000}{999e^{-rt} + 1}.$$

Now, using the information $P(4) = 50$, we determine r from

$$\begin{aligned} 50 &= \frac{1000}{999e^{-4r} + 1}, \\ e^{-4r} &= \frac{19}{999}, \\ -4r &= \ln\left(\frac{19}{999}\right), \\ r &= \frac{-1}{4} \ln\left(\frac{19}{999}\right) = 0.9906. \end{aligned}$$

Thus

$$P(t) = \frac{1000}{999e^{-0.9906t} + 1}.$$

Finally,

$$P(6) = \frac{1000}{999e^{-5.9436} + 1} = 276 \text{ students}.$$

Note that the number of infected students $P(t)$ approaches 1000 as t increases.

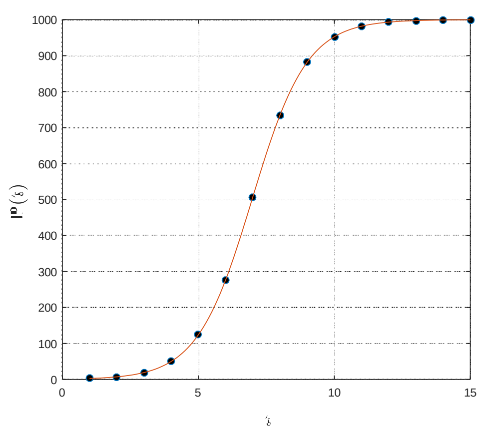


Fig. 3.2

Chapter 4

Linear Second-Order Equations

A second order ordinary differential equation has the form

$$y'' = f(x, y, y'), \tag{4.1}$$

where f is some given function. Usually, we will denote the independent variable by x and the dependent variable by $y = y(x)$. A linear second order ODE has the form

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = a_4(x). \tag{4.2}$$

Of course, if $a_1(x) \neq 0$, we can divide Eq. (4.2) by $a_1(x)$ and thereby obtain the standard form of a linear second order ODE

$$y'' + p(x)y' + q(x)y = g(x). \tag{4.3}$$

We say that Eq. (4.3) is homogeneous if $g(x) = 0$ or nonhomogeneous if $g(x) \neq 0$. In discussing Eq. (4.3) and in trying to solve it, we will restrict ourselves to intervals in which p, q, and g are continuous functions.

We begin our discussion with homogeneous equations, which we will write in the form

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = 0.$$

In this chapter we will concentrate our attention on equations in which the functions a_1, a_2, and a_3 are constants. In this case, Eq. (4.3) becomes

$$ay'' + by' + cy = g(x),$$

where a, b, and c are given constants.

An initial value problem consists of a differential equation such as Eq. (4.1) together with a pair of initial conditions

$$y'' = f(x, y, y'),$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Observe that the initial conditions for a second order equation prescribe not only a particular point (x_0, y_0) through which the graph of the solution must pass, but also the slope y_1 of the graph at that point.

4.1 Fundamental Solutions of Linear Homogeneous Equations

This section is devoted to linear homogeneous second order ODEs of the form

$$y'' + p(x)y' + q(x)y = 0. \tag{4.4}$$

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for linear second order ODEs.

Theorem 4.1 *Consider the IVP*

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(x_0) = y_0, \quad y'(x_0) = y_1. \tag{4.5}$$

If p, q, and g are continuous on an open interval I containing x_0, then there is exactly one solution $y = \varphi(x)$ of this problem, and the solution exists throughout the interval I.

Example 4.1 Find the longest interval in which the solution of the initial value problem

$$x\,(x-4)\,y'' + 3xy' + 4y = 2, \qquad y(3) = 0, \quad y'(3) = -1$$

is certain to exist.

Solution: We first rewrite the DE in the form of Eq. (4.5). To do so, we divide the DE by $x\,(x-4)$ to have

$$y'' + \frac{3}{(x-4)}y' + \frac{4}{x\,(x-4)}y = \frac{2}{x\,(x-4)}.$$

Then $p(x) = \frac{3}{(x-4)}$, $q(x) = \frac{4}{x(x-4)}$, and $g(x) = \frac{4}{x(x-4)}$. The only points of discontinuity of the coefficients are $x = 0$ and $x = 4$. Therefore, the longest open interval, containing the initial point $x = 3$, in which all the coefficients are continuous is $0 < x < 4$. Thus, this is the longest interval in which Theorem 4.1 guarantees that the solution exists.

If y_1 and y_2 are defined on an interval I and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a linear combination of y_1 and y_2. For example, $y = 2\cos x + 7\sin x$ is a linear combination of $y_1 = \cos x$ and $y_2 = \sin x$, with $c_1 = 2$ and $c_2 = 7$.

Let us now assume that y_1 and y_2 are two solutions of Eq. (4.4), then we can generate more solutions by forming linear combinations of y_1 and y_2.

Theorem 4.2 ***(Principle of Superposition)*** *If y_1 and y_2 are two solutions of the DE*

$$y'' + p(x)y' + q(x)y = 0,$$

then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any values of the constants c_1 and c_2.

A special case of the this theorem occurs if either c_1 or c_2 is zero. Then we conclude that any constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.

The next two concepts are basic to the study of linear differential equations.

Definition 4.1 Two functions $f(x)$ and $g(x)$ defined on an open interval I are said to be linearly independent on I provided that neither is a constant multiple of the other. Two functions are said to be linearly dependent on an open interval provided that they are not linearly independent there; that is, one of them is a constant multiple of the other.

According to the above definition, two functions $f(x)$ and $g(x)$ are linearly depependent on I if there is a constant c such that $f(x) = cg(x)$ for all $x \in I$. In this case the quotients $f/g = c$ or $g/f = 1/c$ are constant-valued function on I. We can always determine whether two given functions f and g are linearly dependent on an interval I by:

- If $\frac{f}{g}$ is a constant-valued function on I, then f and g are linearly dependent on I.
- If $\frac{f}{g}$ is not a constant-valued function on I, then f and g are linearly independent on I.

Example 4.2 The pair of functions $\sin x$ and $\cos x$ are linearly independent on any interval. This is clear beacause $\sin x/\cos x = \tan x$ is not a constant-valued function. But the functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent on any interval because

$$f(x) = \sin 2x = 2 \sin x \cos x = 2g(x).$$

Definition 4.2 Given two functions f and g, the Wronskian of f and g is the determinant

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'.$$

We write either $W(f;g)$ or $W(f;g)(x)$, depending on whether we wish to emphasize the two functions or the point x at which the Wronskian is to be evaluated.

For example,

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x (xe^x + e^x) - e^x (xe^x) = e^{2x}$$

and

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

The following theorem gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 4.3 ***(Abel's Theorem)*** *If y_1 and y_2 are solutions of the differential equation*

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)$ is given by

$$W(y_1, y_2) = ce^{-\int p(x)dx},$$

where c is a certain constant that depends on y_1 and y_2, but not on x.

Of course, we know by Abel's Theorem that $W(y_1, y_2)$ is either everywhere zero or nowhere zero in I. The next theorem says that two solutions y_1 and y_2 of Eq. (4.4) are linearly independent on I if and only if $W(y_1, y_2)$ has no zeros on I.

Theorem 4.4 *Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{4.6}$$

on an open interval I on which p and q are continuous. Then

(a) y_1 and y_2 are linearly dependent on I if and only if $W(y_1, y_2)(x) = 0$ for all $x \in I$.
(b) y_1 and y_2 are linearly independent on I if and only if $W(y_1, y_2)(x) \neq 0$ at each $x \in I$.

Proof. Observe first that if y_1 and y_2 are linearly dependent, then $y_1 = k$ y_2 and so

$$W(y_1, y_2)(x) = \begin{vmatrix} ky_2 & y_2 \\ ky_2' & y_2' \end{vmatrix} = 0$$

for all x in I. On the other hand, let $W(y_1, y_2) = 0$ throughout I. Choose any point x_0 in I; then necessarily $W(y_1, y_2)(x_0) = 0$. Consequently, the system of equations

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0, \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0, \end{aligned} \tag{4.7}$$

for c_1 and c_2 has a nontrivial solution. Using these values of c_1 and c_2, let

$$\varphi(x) = c_1 y_1(x) + c_2 y_2(x).$$

Then φ is a solution of Eq. (4.6), and by Eqs. (4.7) φ also satisfies the initial conditions

$$\varphi(x_0) = 0, \qquad \varphi'(x_0) = 0.$$

Therefore, by Theorem 4.1 $\varphi(x) = 0$ for all x in I. Since

$$\varphi(x) = c_1 y_1(x) + c_2 y_2(x) = 0$$

with c_1 and c_2 not both zero, this means that y_1 and y_1 are linearly dependent. □

Example 4.3 Show that the functions e^{3x} and e^{-2x} are linearly independent on any interval.

Solution: We calculate the Wronskian of the given two funcions:

$$W(e^{3x}, e^{-2x}) = \begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -5e^{x}.$$

Since $W(e^{3x}, e^{-2x})(x) \neq 0$ at each $x \in \mathbb{R}$, then by Theorem 4.4 the functions e^{3x} and e^{-2x} are linearly independent on any interval.

The expression

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with arbitrary constant coefficients is called the **general solution** of Eq. (4.4) if it contains all solutions of Eq. (4.4). Moreover, the solutions y_1 and y_2 are said to form a **fundamental set of solutions** of Eq. (4.4).

The following theorem states that any two linearly independent solutions of the second-order homogeneous equation (4.4) form a fundamental set of solutions.

Theorem 4.5 *If y_1 and y_2 are two linearly independent solutions of the homogeneous equation (4.4),*

$$y'' + p(x)y' + q(x)y = 0,$$

with p and q continuous on the open interval I, then

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \qquad x \in I,$$

is the general solution of Eq. (4.4).

Proof. Let $Y(x)$ be any solution of Eq. (4.4). Since y_1 and y_2 are linearly independent, we can find a point x_0 where $W(y_1, y_2)(x_0) \neq 0$. Then evaluate Y and Y' at this point and call these values Y_0 and Y_1, respectively; thus

$$Y_0 = Y(x_0), \qquad Y_1 = Y'(x_0).$$

Next, consider the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = Y_0, \qquad y'(x_0) = Y_1.$$

The function Y is certainly a solution of this initial value problem. On the other hand, since $W(y_1, y_2)(x_0) \neq 0$, it is possible to choose c_1 and c_1 so that $y(x) = c_1y_1(x)+c_2y_2(x)$ is also a solution of the initial value problem. In fact, the initial conditions require c_1 and c_1 to satisfy the equations

$$c_1y_1(x_0) + c_2y_2(x_0) = Y_0,$$
$$c_1y_1'(x_0) + c_2y_2'(x_0) = Y_1.$$

Upon solving this system for c_1 and c_1, we find that the proper values of c_1 and c_1 are given by

$$c_1 = \frac{\begin{vmatrix} Y_0 & y_2(x_0) \\ Y_1 & y_2'(x_0) \end{vmatrix}}{W(y_1, y_2)(x_0)}, \qquad c_1 = \frac{\begin{vmatrix} y_1(x_0) & Y_0 \\ y_1'(x_0) & Y_1 \end{vmatrix}}{W(y_1, y_2)(x_0)}.$$

The uniqueness part of Theorem 4.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus

$$Y(x) = c_1y_1(x) + c_2y_2(x).$$

Finally, since Y is an arbitrary solution of Eq. (4.4), it follows that every solution of this equation is included in this family. This completes the proof.

□

Example 4.4 We can easily verify that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$ are two solutions of the DE

$$y'' - 4y = 0.$$

The Wronskian of these solutions is

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4 \neq 0,$$

so they are linearly independent and form a fundamental set of solutions. The general solution of the given DE is

$$y(x) = c_1e^{2x} + c_2e^{-2x}.$$

Example 4.5 Find a pair of linearly independent solutions of the form $y = x^r$ to the following DE for $x > 0$

$$3x^2y'' + 11xy' - 3y = 0, \qquad x > 0,$$

then write it's general solution.

Solution: Inserting $y = x^r$ yields,

$$3r^2 + 8r - 3 = 0,$$

whose roots $r = 1/3$ and $r = -3$ produce the solutions

$$y_1(x) = x^{1/3} \quad \text{and} \quad y_2(x) = x^{-3}.$$

We calculate the Wronskian of y_1 and y_2:

$$W(y_1, y_2)(x) = \begin{vmatrix} x^{1/3} & x^{-3} \\ \frac{1}{3}x^{-2/3} & -3x^{-4} \end{vmatrix} = -\frac{10}{3x^{11/3}} \neq 0.$$

Since W is nonzero for every value of $x > 0$, then y_1 and y_2 form a fundamental set of solutions. Consequently, the general solution is given by

$$y(x) = c_1 x^{1/3} + c_2 x^{-3}.$$

Example 4.6 The functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By inspection the solutions are linearly independent on $(-\infty, \infty)$. This fact can be corroborated by observing that the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently, $y(x) = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on the interval. The function $y = 4\sinh 3x - 5e^{-3x}$ is also a solution of the differential equation. (Verify this.) In view of Theorem 4.5 we must be able to obtain this solution from the general solution $y(x) = c_1 e^{3x} + c_2 e^{-3x}$. Observe that

$$\begin{aligned} y &= 4\sinh 3x - 5e^{-3x} \\ &= 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x} \\ &= 2e^{3x} - 7e^{-3x}. \end{aligned}$$

Therefore, $y(x) = 4\sinh 3x - 5e^{-3x} = c_1 e^{3x} + c_2 e^{-3x}$ with $c_1 = 2$ and $c_2 = -7$.

4.2 Reduction of Order

Let $y_1(x)$ be a known nonzero solution to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0, \tag{4.8}$$

with p and q continuous on the open interval I. To write the general solution to Eq. (4.8), we need a second linearly independent solution $y_2(x)$. The Method of Reduction of Order is a well-known method to obtain such second solution, $y_2(x)$.

Reduction of Order method assumes that

$$y_2(x) = v(x)y_1(x).$$

We can find that

$$y_2'(x) = v(x)y_1'(x) + v'(x)y_1(x),$$
$$y_2''(x) = v(x)y_1''(x) + 2v'(x)y_1'(x) + v''(x)y_1(x).$$

Substituting these expressions for y_2, y_2', and y_2'' in Eq. (4.8) and rearranging terms, we obtain

$$\left(y_1'' + p(x)y_1' + q(x)y_1\right)v + y_1v'' + \left(2y_1' + p(x)y_1\right)v' = 0. \tag{4.9}$$

Since y_1 is a solution to Eq. (4.8) then $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and Eq. (4.9) reduces to

$$y_1v'' + \left(2y_1' + p(x)y_1\right)v' = 0.$$

Rearranging the last equation we get

$$\frac{v''}{v'} = -\left(\frac{2y_1'}{y_1} + p(x)\right). \tag{4.10}$$

We now integrate the equation in (4.10) to obtain

$$\ln v' = -2\ln y_1 - \int p(x)dx.$$

Exponentiating both sides yields

$$v' = e^{-2\ln y_1 - \int p(x)dx} = e^{-2\ln y_1}e^{-\int p(x)dx} = \frac{e^{-\int p(x)dx}}{y_1^2}.$$

An integration with respect to x now gives

$$v(x) = \int \left(\frac{e^{-\int p(x)dx}}{y_1^2}\right)dx. \tag{4.11}$$

This formula provides a second solution $y_2(x) = v(x)y_1(x)$ of Eq. Eq. (4.8) on any interval where $y_1(x)$ is never zero. Note that because an exponential function never vanishes, $y_2(x)$ is a nonconstant multiple of $y_1(x)$, so $y_1(x)$ and $y_2(x)$ are linearly independent solutions.

Example 4.7 Given that $y_1(x) = x^2$ is a solution of

$$x^2y'' - 3xy' + 4y = 0, \qquad x > 0,$$

find a second linearly independent solution.

Solution: We first divide the equation by its leading coefficient x^2 to get the standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0.$$

Thus we have $p(x) = -\frac{3}{x}$. Reduction of Order method will be used to obtain a second linearly independent solution $y_2(x)$. The method assumes that $y_2(x) = x^2v(x)$ where $v(x)$ is given by formula (4.11) as

$$\begin{aligned} v(x) &= \int \left(\frac{e^{\int \frac{3}{x} dx}}{x^4} \right) dx \\ &= \int \left(\frac{e^{3 \ln x}}{x^4} \right) dx \\ &= \int \frac{1}{x} dx \\ &= \ln x \end{aligned}$$

for $x > 0$. Thus our equation has the two independent solutions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ for $x > 0$.

Example 4.8 Given that $y_1(x) = e^x$ is a solution to the DE

$$xy'' - (1 + x)\, y' + y = 0, \qquad x > 0.$$

Write the general solution to this DE.

Solution: Before we can apply the Reduction of Order formula to find a second solution, we must first divide the equation by its leading coefficient x to get the standard form

$$y'' - \left(\frac{1}{x} + 1 \right) y' + \frac{1}{x}y = 0.$$

Thus we have $p(x) = -\left(\frac{1}{x}+1\right)$. The Reduction of Order formula in (4.11) yields the second linearly independent solution $y_2(x) = e^x v(x)$ with

$$\begin{aligned} v(x) &= \int \left(\frac{e^{\int\left(\frac{1}{x}+1\right)dx}}{e^{2x}}\right) dx \\ &= \int \left(\frac{e^{\ln x + x}}{e^{2x}}\right) dx \\ &= \int \left(\frac{xe^x}{e^{2x}}\right) dx \\ &= \int xe^{-x} dx \qquad\qquad \text{(Integration by parts)} \\ &= -xe^{-x} - e^{-x} \end{aligned}$$

for $x > 0$. It follows that $y_2(x) = -(x+1)e^{-x}e^x = -(x+1)$ and consequently the general solution is given by

$$y(x) = c_1 e^x + c_2(x+1).$$

4.3 Homogeneous Equations with Constant Coefficients

Consider the second order linear ODE with constant coefficients

$$ay'' + by' + cy = 0, \tag{4.12}$$

where a, b, and c are given constants. To write the general solution for Eq. (4.12) we need two linearly independent solutions. To this end, we try to find exponential solutions of the form

$$y(x) = e^{rx}.$$

Then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$. If we substitute in Eq. (4.12) we find the result to be

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0;$$

that is,

$$\left(ar^2 + br + c\right) e^{rx} = 0.$$

Because e^{rx} is never zero, we see that $y = e^{rx}$ will be a solution of Eq. (4.12) precisely when r is a root of the quadratic equation

$$ar^2 + br + c = 0. \tag{4.13}$$

This equation is called the characteristic equation or auxiliary equation of the differential equation in (4.12). Our problem, then, is reduced to the solution of this purely algebraic equation. According to the fundamental theorem of algebra, every nth-degree polynomial has n zeros, though not necessarily distinct and not necessarily real.

The roots of the auxiliary equation can be obtained using the general formula

$$r = \frac{-b \pm \sqrt{\Delta}}{2a},$$

where $\Delta = b^2 - 4ac$ is called the discriminant of the quadratic equation (4.13).

Upon solving the the auxiliary equation (4.13), we face one of the following three cases:

- Case $\Delta > 0$: The roots of the characteristic equation (4.13) are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ are two solutions of Eq. (4.12). The Wronskian of y_1 and y_2 is

$$W(e^{r_1 x}, e^{r_2 x}) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1 + r_2)x}.$$

  Since the roots are distinct then $r_2 - r_1 \neq 0$ and consequently the Wronskian is not zero. So y_1 and y_2 form a fundamental set of solutions and as a result the general solution of Eq. (4.12) is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

- Case $\Delta = 0$: The roots of the characteristic equation (4.13) are repeated real roots, let them be denoted by $r_1 = r_2 = -\frac{b}{2a}$. The difficulty is immediately apparent; both roots yield the same solution

$$y_1(x) = e^{-\frac{b}{2a}x}$$

  of the differential equation (4.12). To find a second linearly independent solution the Reduction of Order method will be used. Starting with $y_2(x) = v(x)e^{r_1 x}$, we have from the Reduction of Order formula in (4.11) that

$$v(x) = \int \left(\frac{e^{-\int \frac{b}{a} dx}}{e^{-\frac{b}{a}x}} \right) dx = \int 1 dx = x.$$

Therefore $y_2(x) = xe^{-\frac{b}{2a}x}$. The Wronskian of these two solutions $y_1(x) = e^{-\frac{b}{2a}x}$ and $y_2(x) = xe^{-\frac{b}{2a}x}$ is

$$W(y_1, y_2) = \begin{vmatrix} e^{-\frac{b}{2a}x} & xe^{-\frac{b}{2a}x} \\ -\frac{b}{2a}e^{-\frac{b}{2a}x} & -\frac{b}{2a}xe^{-\frac{b}{2a}x} + e^{-\frac{b}{2a}x} \end{vmatrix} = e^{-\frac{b}{a}x}.$$

Since $W(y_1, y_2)$ is never zero, the solutions y_1 and y_2 are a fundamental set of solutions. Further, the general solution of Eq. (4.12) is

$$y(x) = c_1 e^{-\frac{b}{2a}x} + c_2 xe^{-\frac{b}{2a}x}.$$

- Case $\Delta < 0$: The characteristic equation (4.13) has conjugate complex roots $r = \alpha \pm \beta i$, where α and $\beta > 0$ are real and $i^2 = -1$. In this case the two solutions $y_1 = e^{(\alpha+\beta i)x}$ and $y_2 = e^{(\alpha-\beta i)x}$ are complex-valued functions, whereas in practice we would prefer to have real-valued solutions. To this end we use Euler's formula:

$$e^{iz} = \cos z + i \sin z,$$

where z is any real number. It follows from this formula that

$$\begin{aligned} e^{i\beta x} &= \cos \beta x + i \sin \beta x, \\ e^{-i\beta x} &= \cos \beta x - i \sin \beta x. \end{aligned} \tag{4.14}$$

Note that by first adding and then subtracting the two equations in (4.14), we obtain, respectively,

$$\begin{aligned} e^{i\beta x} + e^{-i\beta x} &= 2 \cos \beta x, \\ e^{i\beta x} - e^{-i\beta x} &= 2i \sin \beta x. \end{aligned} \tag{4.15}$$

The superposition principle states that any linear combination of y_1 and y_2 is also a solution. In particular, let us form the sum and then the difference of y_1 and y_2. We have

$$\begin{aligned} y_1 + y_2 &= e^{(\alpha+\beta i)x} + e^{(\alpha-\beta i)x} = e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) = 2e^{\alpha x} \cos \beta x, \\ y_1 - y_2 &= e^{(\alpha+\beta i)x} - e^{(\alpha-\beta i)x} = e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right) = 2ie^{\alpha x} \sin \beta x. \end{aligned}$$

Hence, neglecting the constant multipliers 2 and $2i$, respectively, we have obtained a pair of real-valued solutions

$$u(x) = e^{\alpha x} \cos \beta x, \qquad v(x) = e^{\alpha x} \sin \beta x.$$

We can easily show that the Wronskian of u and v is

$$W(u,v) = \beta e^{2\alpha x}.$$

Thus, as long as $\beta \neq 0$, the Wronskian W is not zero, so u and v form a fundamental set of solutions. Consequently, the general solution is

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$

Example 4.9 Solve the following differential equations.
(a) $12y''-5y'-2y = 0$ (b) $y''-10y'+25y = 0$ (c) $2y''-3y'+4y = 0$
Solution:
(a) The auxiliary equation is $12r^2 - 5r - 2 = 0$. The discriminant of this equation is

$$\Delta = b^2 - 4ac = 25 - 4(12)(-2) = 121 > 0$$

Thus we have two distinct real roots

$$r_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{5-11}{24} = -\frac{1}{4},$$
$$r_2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{5+11}{24} = \frac{2}{3}.$$

Therefore the general solution of the differential equation is

$$y(x) = c_1 e^{-x/4} + c_2 e^{2x/3}.$$

(b) The auxiliary equation is $r^2 - 10r + 25 = 0$. Since the discriminant $\Delta = 10^2 - 4(1)(25) = 0$, then we have repeated roots

$$r_1 = r_2 = \frac{-b}{2a} = 5.$$

Consequently, the given differential equation has the general solution

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}.$$

(c) The auxiliary equation is $2r^2 - 3r + 4 = 0$ and its roots are complex because $\Delta = 3^2 - 4(2)(4) = -23 < 0$. Using the general formula, the roots are

$$r = \frac{3 \pm \sqrt{-23}}{4} = \frac{3 \pm i\sqrt{23}}{4} = \frac{3}{4} \pm \frac{\sqrt{23}}{4} i.$$

Thus $\alpha = \frac{3}{4}$ and $\beta = \frac{\sqrt{23}}{4}$, so the general solution of the DE is

$$y(x) = c_1 e^{\frac{3}{4}x} \cos\left(\frac{\sqrt{23}}{4}x\right) + c_2 e^{\frac{3}{4}x} \sin\left(\frac{\sqrt{23}}{4}x\right).$$

Example 4.10 Find the solution of the initial value problem

$$y'' + 2y' + 4y = 0, \qquad y(0) = 0, \qquad y'(0) = 2. \tag{4.16}$$

Solution: The auxiliary equation associated with the DE is $r^2+2r+4 = 0$. Since $\Delta = 2^2 - 4(1)(4) = -12 < 0$, the auxiliary equation has the complex roots

$$r = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i.$$

Thus $\alpha = -1$ and $\beta = \sqrt{3}$, so the general solution of the differential equation is

$$y(x) = c_1 e^{-x} \cos\left(\sqrt{3}x\right) + c_2 e^{-x} \sin\left(\sqrt{3}x\right). \tag{4.17}$$

Applying the first initial condition $y(0) = 0$ gives

$$y(0) = c_1 = 0$$

It follows that the solution can be expressed as

$$y(x) = c_2 e^{-x} \sin\left(\sqrt{3}x\right)$$

For the second initial condition we must differentiate Eq. (4.17);

$$y'(x) = \sqrt{3}c_2 e^{-x} \cos\left(\sqrt{3}x\right) - c_2 e^{-x} \sin\left(\sqrt{3}x\right)$$

and then set $x = 0$. In this way we find that

$$\sqrt{3}c_2 = 2,$$

from which $c_2 = \frac{2}{\sqrt{3}}$. Using these values of c_1 and c_1 in Eq. (4.17), we obtain

$$y(x) = \frac{2}{\sqrt{3}} e^{-x} \sin\left(\sqrt{3}x\right)$$

as the solution of the initial value problem (4.16).

4.4 Cauchy-Euler Equation

A second-order linear differential equation that can be expressed in the form

$$ax^2y'' + bxy' + cy = 0, \qquad x > 0, \tag{4.18}$$

where a, b, and c are constants, is called a Cauchy-Euler equation.

To solve a Cauchy-Euler equation, we assume solutions of the form

$$y(x) = x^r.$$

Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting the values of y, y' and y'' in the Cauchy-Euler equation (4.18) yields

$$[ar(r-1) + br + c]\, x^r = 0.$$

Since $x^r \neq 0$ for $x > 0$, then

$$ar(r-1) + br + c = 0.$$

Rearranging the last equation gives the quadratic equation

$$ar^2 + (b-a)r + c = 0, \tag{4.19}$$

which is called the auxiliary equation associated with the Cauchy-Euler equation (4.18).

Upon solving the the auxiliary equation (4.19), we face one of the following three cases:

- The roots of the characteristic equation (4.19) are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are two solutions of Eq. (4.18). The Wronskian of y_1 and y_2 is

$$W(x^{r_1}, x^{r2}) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1-1} & r_2 x^{r_2-1} \end{vmatrix} = (r_2 - r_1)\, x^{r_1+r_2-1}.$$

  Since $x > 0$ and the roots are distinct then $x^{r_1+r_2-1} \neq 0$ and $r_2 - r_1 \neq 0$. It follows that the Wronskian is not zero and so y_1 and y_2 form a fundamental set of solutions. Consequently the general solution of Eq. (4.18) is given by

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}.$$

- The roots of the characteristic equation (4.19) are repeated real roots, let them be denoted by $r_1 = r_2 = -\frac{b-a}{2a}$. The difficulty is immediately apparent; both roots yield the same solution

$$y_1(x) = x^{-\frac{b-a}{2a}}$$

of the Cauchy-Euler equation (4.18). We will employ the Reduction of Order method to obtain a second linearly independent solution. Starting with $y_2(x) = v(x)x^{-\frac{b-a}{2a}}$ and noticing that $p(x) = \frac{b}{ax}$, we have from the Reduction of Order formula in (4.11) that

$$\begin{aligned} v(x) &= \int \left(\frac{e^{-\int \frac{b}{ax}dx}}{x^{-\frac{b-a}{a}}} \right) dx = \int \left(\frac{e^{-\frac{b}{a}\ln x}}{x^{-\frac{b-a}{a}}} \right) dx \\ &= \int \left(\frac{x^{-\frac{b}{a}}}{x^{-\frac{b}{a}}x} \right) dx = \int \frac{1}{x} dx = \ln x, \end{aligned}$$

for $x > 0$. Therefore $y_2(x) = x^{-\frac{b-a}{2a}} \ln x$. The Wronskian of these two solutions is

$$W(y_1, y_2) = \begin{vmatrix} x^{-\frac{b-a}{2a}} & x^{-\frac{b-a}{2a}} \ln x \\ -\frac{b-a}{2a}x^{-\frac{b-a}{2a}-1} & x^{-\frac{b-a}{2a}-1} - \frac{b-a}{2a}x^{-\frac{b-a}{2a}-1} \ln x \end{vmatrix} = x^{-\frac{b-a}{a}-1}.$$

Thus, as long as $x > 0$, the Wronskian W is not zero, so y_1 and y_2 form a fundamental set of solutions. Further, the general solution of Eq. (4.18) is

$$y(x) = c_1 x^{-\frac{b-a}{2a}} + c_2 x^{-\frac{b-a}{2a}} \ln x.$$

- The characteristic equation (4.19) has conjugate complex roots $r = \alpha \pm \beta i$, where α and $\beta > 0$ are real and $i^2 = -1$. In this case the two solutions $y_1 = x^{(\alpha+\beta i)}$ and $y_2 = x^{(\alpha-\beta i)}$ are complex-valued functions, whereas in practice we would prefer to have real-valued solutions. The superposition principle states that any linear combination of y_1 and y_2 is also a solution. In particular, let us form the sum and then the difference of y_1 and y_2. With the help of the identities (4.15)

$$\begin{aligned} e^{i\beta x} + e^{-i\beta x} &= 2\cos\beta x, \\ e^{i\beta x} - e^{-i\beta x} &= 2i\sin\beta x, \end{aligned}$$

we have for $x > 0$ that

$$\begin{aligned} y_1 + y_2 &= x^{(\alpha+\beta i)} + x^{(\alpha-\beta i)} = e^{(\alpha+\beta i)\ln x} + e^{(\alpha-\beta i)\ln x} \\ &= e^{\alpha \ln x}\left(e^{i\beta\ln x} + e^{-i\beta\ln x}\right) = 2x^\alpha \cos\left(\beta \ln x\right), \end{aligned}$$

and

$$\begin{aligned} y_1 - y_2 &= x^{(\alpha+\beta i)} - x^{(\alpha-\beta i)} = e^{(\alpha+\beta i)\ln x} - e^{(\alpha-\beta i)\ln x} \\ &= e^{\alpha \ln x}\left(e^{i\beta\ln x} - e^{-i\beta\ln x}\right) = 2ix^\alpha \sin\left(\beta \ln x\right). \end{aligned}$$

Hence, neglecting the constant multipliers 2 and $2i$, respectively, we have obtained a pair of real-valued solutions

$$u(x) = x^\alpha \cos\left(\beta \ln x\right), \qquad v(x) = x^\alpha \sin\left(\beta \ln x\right).$$

A straightforward calculation shows that

$$W(u,v)(x) = \beta x^{2\alpha-1}.$$

Thus, as long as $\beta \neq 0$, the Wronskian W is not zero for $x > 0$, so u and v form a fundamental set of solutions. Consequently, the general solution is

$$y(x) = c_1 x^\alpha \cos\left(\beta \ln x\right) + c_2 x^\alpha \sin\left(\beta \ln x\right).$$

Example 4.11 Find the general solution to the differential equation

$$2x^2y'' + 3xy' - y = 0, \qquad x > 0. \tag{4.20}$$

Solution: This is a Cauchy-Euler equation with $a = 2$, $b = 3$, and $c = -1$. We seek solutions of the form $y = x^r$. Substituting in Eq. (4.20) gives that r must satisfy the auxiliary equation

$$\begin{aligned} ar^2 + (b-a)r + c &= 0, \\ 2r^2 + r - 1 &= 0. \end{aligned}$$

For this quadratic equation $\Delta = 1^2 - 4(2)(-1) = 9 > 0$, and so it has two distinct real roots $r_1 = \frac{1}{2}$ and $r_2 = -1$. This produces the solutions $y_1(x) = x^{1/2}$ and $y_2(x) = x^{-1}$. Thus the general solution of Eq. (4.20) is

$$y(x) = c_1\sqrt{x} + c_2\frac{1}{x}.$$

Example 4.12 Solve the Cauchy-Euler equation

$$x^2y'' - 3xy' + 4y = 0, \qquad x > 0. \tag{4.21}$$

Solution: Substituting $y = x^r$ in Eq. (4.21) gives the auxiliary equation

$$ar^2 + (b-a)r + c = 0,$$
$$r^2 - 4r + 4 = 0.$$

The auxiliary equation has repeated roots $r_1 = r_2 = 2$. This gives the two solutions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$, so the general solution of Eq. (4.21) is

$$y(x) = c_1x^2 + c_2x^2 \ln x, \qquad x > 0.$$

Example 4.13 Find a general solution for

$$x^2y'' - 3xy' + 6y = 0, \qquad x > 0. \tag{4.22}$$

Solution: We seek solutions of the form $y = x^r$. The auxiliary equation associated with this Cauchy-Euler equation is

$$r^2 - 4r + 6 = 0,$$

with the complex roots $r = 2 \pm \sqrt{2}i$. Using that $\alpha = 2$ and $\beta = \sqrt{2}$, this produces the real solutions $y_1(x) = x^2 \cos\left(\sqrt{2}\ln x\right)$ and $y_2(x) = x^2 \sin\left(\sqrt{2}\ln x\right)$. Hence the general solution of Eq. (4.22) is

$$y(x) = c_1x^2 \cos\left(\sqrt{2}\ln x\right) + c_2x^2 \sin\left(\sqrt{2}\ln x\right), \qquad x > 0.$$

4.5 Nonhomogeneous Equations

We now return to the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x), \tag{4.23}$$

where p, q, and g are given (continuous) functions on the open interval I and $g(x) \neq 0$. Putting $g(x) = 0$ in Eq. (4.23) we obtain the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{4.24}$$

Eq. (4.24) is called the homogeneous equation corresponding to Eq. (4.23). The following two results describe the structure of solutions of the nonhomogeneous equation (4.23) and provide a basis for constructing its general solution.

Theorem 4.6 *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (4.23), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (4.23). If, in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (4.24), then*

$$Y_1(x) - Y_2(x) = c_1 y_1(x) + c_2 y_2(x) \tag{4.25}$$

where c_1 and c_2 are certain constants.

The next theorem shows how to find the general solution of (4.23) if we know one solution y_p of (4.23) and a fundamental set of solutions of (4.24). We call y_p a particular solution of (4.23); it can be any solution that we can find, one way or another.

Theorem 4.7 *The general solution of the nonhomogeneous equation (4.23) can be written in the form*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x), \tag{4.26}$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation (4.24), c_1 and c_2 are arbitrary constants, and y_p is some particular solution of the nonhomogeneous equation (4.23).

Proof. Suppose we know some specific solution, y_p, of the nonhomogeneous equation (4.23). Let $\varphi(x)$ be an arbitrary solution of Eq. (4.23). From Eq. (4.25) we thereby obtain

$$\varphi(x) - y_p(x) = c_1 y_1(x) + c_2 y_2(x)$$

which is equivalent to

$$\varphi(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Since φ is an arbitrary solution of Eq. (4.23), the expression on the right side of Eq. (4.26) includes all solutions of Eq. (4.23). □

Thus it is natural to call the solution in (4.26) the general solution of Eq. (4.23). In In summary, to solve the nonhomogeneous equation (4.23), we must do three things:

- Find the general solution $y_h(x) = c_1y_1(x)+c_2y_2(x)$ of the corresponding homogeneous equation. This solution is frequently called the complementary solution and may be denoted by $y_c(t)$.
- Find some single particular solution $y_p(x)$ of the nonhomogeneous equation.
- Add together the functions found in the two preceding steps to get the general solution $y(x) = y_h(x) + y_p(x)$.

We have already discussed how to find $y_h(x)$, at least homogeneous equations with constant coefficients and Cauchy-Euler equations. Therefore, in the coming two sections, we will focus on finding a particular solution $y_p(x)$ of the nonhomogeneous equation (4.23). There are two methods that we will discuss. They are known as the method of undetermined coefficients and the method of variation of parameters, respectively. Each has some advantages and some possible shortcomings.

4.6 Method of Undetermined Coefficients

The method of undetermined coefficients is usually used to find a particular solution, $y_p(x)$, for nonhomogeneous equations of the form

$$ay'' + by' + cy = g(x). \tag{4.27}$$

The method requires that we make an initial assumption about the form of the particular solution $y_p(x)$, but with the coefficients left unspecified. We then substitute the assumed expression into Eq. (4.27) and attempt to determine the coefficients so as to satisfy that equation. If we are successful, then we have found a solution of the differential equation (4.27) and can use it for the particular solution $y_p(x)$. If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again. The method of undetermined coefficients is restricted to a relatively small class of functions, $g(x)$. In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines.

The next theorem describes the superposition principle. It extends the applicability of the method of undetermined coefficients and enables us to solve initial value problems for nonhomogeneous differential equations.

Theorem 4.8 ***(Superposition Principle)*** *If y_1 is a solution to the* *differential equation*

$$ay'' + by' + cy = g_1(x),$$

and y_2 *is a solution to*

$$ay'' + by' + cy = g_2(x),$$

then for any constants k_1 *and* k_2*, the function* $k_1y_1 + k_2y_2$ *is a solution to the differential equation*

$$ay'' + by' + cy = k_1g_1(x) + k_2g_2(x).$$

Example 4.14 Given that $y_1(x) = (1/4)\sin 2x$ is a solution to $y''+2y'+4y = \cos 2x$ and that $y_2(x) = x/4 - 1/8$ is a solution to $y'' + 2y' + 4y = x$, use the superposition principle to find a solution to the DE $y'' + 2y' + 4y = 11x - 12\cos 2x$.

Solution: Let $g_1(x) := \cos 2x$ and $g_2(x) := x$. Then $y_1(x) = (1/4)\sin 2x$ is a solution to

$$y'' + 2y' + 4y = g_1(x)$$

and $y_2(x) = x/4 - 1/8$ is a solution to

$$y'' + 2y' + 4y = g_2(x).$$

We can express $11x-12\cos 2x = 11g_2(x)-12g_1(x)$. So, by the superposition principle, the function

$$y(x) = 11y_2(x) - 12y_1(x) = 11x/4 - 11/8 - 3\sin 2x$$

is a solution to the given equation $y'' + 2y' + 4y = 11x - 12\cos 2x$.

We now summarize the steps involved in finding the general solution of Eq. (4.27) using the method of undetermined coefficients:

(1) Find the general solution of the corresponding homogeneous equation, $y_h(x)$.
(2) Make sure that the function $g(x)$ in Eq. (4.27) belongs to the class of functions: exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use another method like the method of variation of parameters (discussed in the next section).

(3) If $g(x) = g_1(x) + \cdots + g_n(x)$, that is, if $g(x)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $g_1(x), \cdots, g_n(x)$. The ith subproblem consists of the equation

$$ay'' + by' + cy = g_i(x),$$

where i runs from 1 to n.

(4) For the ith subproblem assume a particular solution $y_{p_i}(x)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $y_{p_i}(x)$ with the solutions of the homogeneous equation (found in step 1), then multiply $y_{p_i}(x)$ by x, or (if necessary) by x^2, so as to remove the duplication. See the table below.

(5) Find a particular solution $y_{p_i}(x)$ for each of the subproblems. Then the sum

$$y_p(x) = y_{p_1}(x) + \cdots + y_{p_n}(x)$$

is a particular solution of the full nonhomogeneous equation (4.27).

(6) The general solution of the nonhomogeneous equation (4.27) is

$$y(x) = y_h(x) + y_p(x).$$

$g(x)$	$y_p(x)$
$P_n(x)$	$x^s\left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)$
$P_n(x)e^{ax}$	$x^s\left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)e^{ax}$
$P_n(x)\sin bx$	$x^s[\left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\sin bx$ $+ \left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\cos bx]$
$P_n(x)\cos bx$	$x^s[\left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\sin bx$ $+ \left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\cos bx]$
$e^{ax}P_n(x)\sin bx$	$x^se^{ax}[\left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\sin bx$ $+ \left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\cos bx]$
$e^{ax}P_n(x)\cos bx$	$x^se^{ax}[\left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\sin bx$ $+ \left(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\right)\cos bx]$

Example 4.15 Determine the form of a particular solution for the given differential equations. Do not solve.

(a) $y'' - 2y' - 3y = 3e^{2x}$
The auxiliary equation in this problem is $r^2 - 2r - 3 = 0$ with roots $r_1 = 3$, $r_2 = -1$. The method of undetermined coefficients yields

$$y_p(x) = Ae^{2x}.$$

(b) $y'' - y' = 3x^2 - 5$
The auxiliary equation of the corresponding homogeneous equation in this problem is $r^2 - r = 0$ with roots $r_1 = 0$, $r_2 = 1$. Thus,

$$y_h(x) = c_1 + c_2e^x.$$

Our initial choice for a particular solution is

$$y_p(x) = x^s \left(A_2x^2 + A_1x + A_0\right),$$

but since a constant is a solution of the homogeneous equation, we choose $s = 1$. Thus

$$y_p(x) = x \left(A_2x^2 + A_1x + A_0\right).$$

(c) $y'' - y = \left(1 + x + x^2\right) e^{2x}$
The auxiliary equation of the corresponding homogeneous equation in this problem is $r^2 - 1 = 0$ with roots $r_1 = 1$, $r_2 = -1$. Thus,

$$y_h(x) = c_1e^x + c_2e^{-x}.$$

Our initial choice for a particular solution is

$$y_p(x) = x^s \left(A_2x^2 + A_1x + A_0\right) e^{2x},$$

but since no term in this form is a solution to the corresponding homogeneous equation, we choose $s = 0$. Thus

$$y_p(x) = \left(A_2x^2 + A_1x + A_0\right) e^{2x}.$$

(d) $y'' + 4y = 7 \sin 2x$
The auxiliary equation of the corresponding homogeneous equation in this problem is $r^2 + 4 = 0$ with complex roots $r = \pm 2i$. Thus,

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x.$$

Our initial choice for a particular solution is

$$y_p(x) = x^s\left(A_1 \sin 2x + A_0 \cos 2x\right),$$

but since $\sin 2x$ is a solution of the homogeneous equation, we choose $s = 1$. Thus $y_p(x) = x\left(A_1 \sin 2x + A_0 \cos 2x\right)$.

(e) $y'' - 3y' - 4y = 3e^{-x} + 2\sin x - 8e^x \cos 2x$
The auxiliary equation of the corresponding homogeneous equation in this problem is $r^2 - 3r - 4 = 0$ with roots $r_1 = -1$, $r_2 = 4$. Thus,

$$y_h(x) = c_1 e^{-x} + c_2 e^{4x}.$$

By the superposition principle, we can write a particular solution as the sum of particular solutions of the differential equations

$$y''-3y'-4y = 3e^{-x}, \quad y''-3y'-4y = 2\sin x, \quad y''-3y'-4y = -8e^x \cos 2x.$$

Our initial choice for a particular solution of the first equation is $y_{p_1}(x) = x^s\left(A_0 e^{-x}\right)$, but since e^{-x} is a solution of the homogeneous equation, we choose $s = 1$. Thus $y_{p_1}(x) = A_0 x e^{-x}$. For the second equation our initial choice is $y_{p_2}(x) = x^s\left(A_1 \sin x + A_2 \cos x\right)$. Since $\sin x$ and $\cos x$ are not solutions of the homogeneous equation, we choose $s = 0$. Thus $y_{p_2}(x) = A_1 \sin x + A_2 \cos x$. Finally, for the third equation, $y_{p_3}(x) = \left(A_3 \sin 2x + A_4 \cos 2x\right) e^x$. Therefore a particular solution of the nonhomogeneous equation is their sum, namely,

$$\begin{aligned} y_p(x) &= y_{p_1}(x) + y_{p_2}(x) + y_{p_3}(x) \\ &= A_0 x e^{-x} + A_1 \sin x + A_2 \cos x + A_3 e^x \sin 2x + A_4 e^x \cos 2x. \end{aligned}$$

(f) $y'' + 2y' + 2y = 9e^{-x}\left(x^2 - 3\right)\sin x$
The auxiliary equation of the corresponding homogeneous equation is $r^2 + 2r + 2 = 0$ with complex roots $r = -1 \pm i$. Thus,

$$y_h(x) = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x.$$

Our initial choice for a particular solution is

$$y_p(x) = x^s e^{-x}\left[\left(A_0 + A_1 x + A_2 x^2\right)\sin x + \left(A_3 + A_4 x + A_5 x^2\right)\cos x\right],$$

but since $e^{-x} \sin x$ is a solution of the homogeneous equation, we choose $s = 1$. Thus

$$y_p(x) = e^{-x}\left(A_0 x + A_1 x^2 + A_2 x^3\right)\sin x + e^{-x}\left(A_3 x + A_4 x^2 + A_5 x^3\right)\cos x.$$

Example 4.16 Find the general solution of the differential equation

$$y'' - 2y' - 3y = 3e^{2x}. \tag{4.28}$$

Solution: We first solve the corresponding homogeneous equation

$$y'' - 2y' - 3y = 0. \tag{4.29}$$

The auxiliary equation for the corresponding homogeneous equation is $r^2 - 2r - 3 = 0$, and it has distinct real roots $r_1 = -1$, $r_2 = 3$. A fundamental set of solutions of Eq. (4.29) is $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$. Hence,

$$y_h(x) = c_1 e^{-x} + c_2 e^{3x}$$

is a general solution to equation (4.29). By the method of undetermined coefficients, a particular solution to the original equation (4.28) has the form

$$y_p(x) = Ae^{2x},$$

where the coefficient A is yet to be determined. To find A we calculate

$$y_p'(x) = 2Ae^{2x}, \qquad y_p''(x) = 4Ae^{2x}$$

and substitute for y, y', and y'' in Eq. (4.28). We obtain

$$\begin{aligned} 4Ae^{2x} - 4Ae^{2x} - 3Ae^{2x} &= 3e^{2x}, \\ -3Ae^{2x} &= 3e^{2x}, \end{aligned}$$

and so $A = -1$. Thus a particular solution is

$$y_p(x) = -e^{2x}.$$

Therefore,

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{3x} - e^{2x}$$

is a general solution to the nonhomogeneous equation (4.28).

Example 4.17 Find the solution to the initial value problem

$$\begin{gathered} y'' + 2y' + y = x^2 + 1 - e^x, \\ y(0) = 0, \quad y'(0) = 2. \end{gathered} \tag{4.30}$$

Solution: The corresponding homogeneous equation is

$$y'' + 2y' + y = 0. \tag{4.31}$$

The auxiliary equation in this problem, $r^2 + 2r + 1 = 0$ has a double root $r_1 = r_2 = -1$. Hence

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$$

is a general solution to the corresponding homogeneous equation (4.31). We now find a particular solution $y_p(x)$ to the original nonhomogeneous equation (4.30). The method of undetermined coefficients yields

$$y_p(x) = A_0 + A_1 x + A_2 x^2 + A_3 e^x.$$

Then,

$$\begin{aligned} y_p'(x) &= A_1 + 2A_2 x + A_3 e^x, \\ y_p''(x) &= 2A_2 + A_3 e^x. \end{aligned}$$

Substituting these expressions into Eq. (4.30) gives

$$\begin{aligned} y_p'' + 2y_p' + y_p &= A_2 x^2 + (A_1 + 4A_2)\, x + (A_0 + 2A_1 + 2A_2) + 4A_3 e^x \\ &= x^2 + 1 - e^x. \end{aligned}$$

Comparing the corresponding coefficients yields

$$A_2 = 1, \quad A_1 + 4A_2 = 0, \quad A_0 + 2A_1 + 2A_2 = 1, \quad 4A_3 = -1.$$

Therefore,

$$A_0 = 7, \quad A_1 = -4, \quad A_2 = 1, \quad A_3 = -\frac{1}{4},$$

so a particular solution of Eq. (4.30) is

$$y_p(x) = 7 - 4x + x^2 - \frac{1}{4} e^x.$$

The general solution is given by

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 x e^{-x} + 7 - 4x + x^2 - \frac{1}{4} e^x.$$

Satisfying the initial conditions yields

$$\begin{aligned} y(0) = 0 &\Longrightarrow c_1 + 7 - \frac{1}{4} = 0 \Longrightarrow c_1 = -\frac{27}{4}, \\ y'(0) = 2 &\Longrightarrow -c_1 + c_2 - 4 - \frac{1}{4} = 2 \Longrightarrow c_2 = \frac{25}{4} + c_1 = -\frac{1}{2}. \end{aligned}$$

Therefore, the solution to the given initial value problem is

$$y(x) = -\frac{27}{4}e^{-x} - \frac{1}{2}xe^{-x} - \frac{1}{4}e^{x} + x^2 - 4x + 7.$$

4.7 Variation of Parameters

Let us point out the kind of situation in which the method of undetermined coefficients cannot be used. Consider, for example, the equation

$$y'' + y = \tan x.$$

The function $g(x) = \tan x$ has infinitely many linearly independent derivatives

$$\sec^2 x, \quad 2\sec^2 x \ \tan x, \quad 4\sec^2 x \ \tan^2 x + 2\sec^4 x, \ldots.$$

Therefore, we do not have available a finite linear combination to use as a trial solution.

We discuss here the method of variation of parameters, which can always be used to find a particular solution of the nonhomogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = g(x), \tag{4.32}$$

provided that we already know the general solution

$$y_h(x) = c_1y_1(x) + c_2y_2(x) \tag{4.33}$$

of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Here, in brief, is the basic idea of the method of variation of parameters. Suppose that we replace the constants, or parameters, c_1 and c_2 in the solution in Eq. (4.33) by functions $u_1(x)$ and $u_2(x)$, respectively. We ask whether it is possible to choose these functions in such a way that the combination

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{4.34}$$

is a solution of the nonhomogeneous equation (4.32). Thus we differentiate Eq. (4.34), obtaining

$$y_p'(x) = u_1'(x)y_1(x) + u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'(x). \tag{4.35}$$

To avoid the appearance of the second derivatives $u_1''(x)$ and $u_2''(x)$, we now set the terms involving $u_1'(x)$ and $u_2'(x)$ in Eq. (4.35) equal to zero; that is, we require that

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0. \tag{4.36}$$

Then Eq. (4.35) becomes

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x)$$

and by differentiating again, we obtain

$$y_p''(x) = u_1'(x)y_1'(x) + u_1(x)y_1''(x) + u_2'(x)y_2'(x) + u_2(x)y_2''(x).$$

Substituting for y_p, y_p', and y_p'' in Eq. (4.32) and then rearranging the terms in the resulting equation we find that

$$u_1(x)\left[y_1'' + p(x)y_1' + q(x)y_1\right] + u_2(x)\left[y_2'' + p(x)y_2' + q(x)y_2\right] \tag{4.37}$$

$$+u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = g(x).$$

But both y_1 and y_2 are solutions to the associaated homogeneous equation, so

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \qquad \text{and} \qquad y_2'' + p(x)y_2' + q(x)y_2 = 0.$$

Therefore Eq. (4.37) reduces to

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = g(x). \tag{4.38}$$

Collecting Eqs. (4.36) and (4.38), we obtain a system of two linear equations in the two derivatives u_1' and u_2':

$$\begin{aligned} u_1'(x)y_1(x) + u_2'(x)y_2(x) &= 0, \\ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) &= g(x). \end{aligned} \tag{4.39}$$

Note that the determinant of coefficients in (4.39) is simply the Wronskian $W(y_1, y_2) \neq 0$. Once we have solved the equations in (4.39) for the derivatives u_1' and u_2', we integrate each to obtain the functions u_1 and u_2, namely,

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}dx, \\ u_2(x) &= \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}dx. \end{aligned} \tag{4.40}$$

Finally, the desired particular solution of Eq. (4.32) is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}dx + y_2(x)\int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}dx. \qquad (4.41)$$

In our first example, we will discuss an equation where the nonhomogeneous term does not belong to any of the families of polynomials, exponential functions, sines, and cosines.

Example 4.18 Find the general solution of the equation

$$y'' + y = \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Solution: This equation has associated homogeneous equation

$$y'' + y = 0.$$

The roots of the auxiliary equation, $r^2 + 1 = 0$, are $r = \pm i$. Therefore, the functions $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are two linearly independent solutions to the corresponding homogeneous equation, and so its general solution is given by

$$y_h(x) = c_1 \cos x + c_2 \sin x.$$

Now we apply the method of variation of parameters to find a particular solution to the original equation. The method assumes that $y_p(x)$ has the form

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x.$$

In this problem $W(\cos x, \sin x) = 1$. We can now use the formulas in (4.40) to find u_1 and u_2:

$$\begin{aligned} u_1(x) &= -\int \sin x \tan x dx = -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx \\ &= \int (\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|, \end{aligned}$$

and

$$u_2(x) = \int \cos x \tan x dx = \int \sin x dx = -\cos x.$$

Therefore,

$$y_p(x) = (\sin x - \ln|\sec x + \tan x|)\cos x + (-\cos x)\sin x,$$

which simplifies to

$$y_p(x) = -\cos x \ln |\sec x + \tan x|$$

Thus, a general solution to the given nonhomogeneous equation is

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln |\sec x + \tan x| .$$

Our second example shows the applicability of the method of variation of parameters to linear equations with variable coefficients. In other words, the method works provided we know a pair of linearly independent solutions to the corresponding homogeneous equation.

Example 4.19 Find the general solution to

$$x^2 y'' - 4xy' + 6y = x^3 + 1, \qquad x > 0. \tag{4.42}$$

Solution: The associated homogeneous equation is the Cauchy-Euler equation

$$x^2 y'' - 4xy' + 6y = 0.$$

Its auxiliary equation, $r^2 - 5r + 6 = 0$, has the roots $r_1 = 2$ and $r_2 = 3$. Hence, a general solution to the homogeneous problem is given by

$$y_h(x) = c_1 x^2 + c_2 x^3.$$

Here,

$$W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4.$$

Using the method of variation of parameters, we look for a particular solution to the given equation in the form

$$y_p(x) = u_1(x) x^2 + u_2(x) x^3.$$

We first divide Eq. (4.42) by its leading coefficients x^2 to get the standard form

$$y'' - \frac{4}{x} y' + \frac{6}{x^2} y = \frac{x^3 + 1}{x^2}$$

and so we have $g(x) = \frac{x^3+1}{x^2}$.

To determine the unknown functions u_1 and u_2, we use the formulas in (4.40):

$$u_1(x) = -\int \frac{x^3 \cdot \frac{x^3+1}{x^2}}{x^4} dx = -\int \left(x^{-3}+1\right) dx = -\left(-\frac{x^{-2}}{2}+x\right),$$

$$u_2(x) = \int \frac{x^2 \cdot \frac{x^3+1}{x^2}}{x^4} dx = \int \left(\frac{1}{x}+x^{-4}\right) dx = \ln x - \frac{x^{-3}}{3}.$$

Hence, a particular solution to Eq. (4.42) is

$$\begin{aligned} y_p(x) &= -\left(-\frac{x^{-2}}{2}+x\right)x^2 + \left(\ln x - \frac{x^{-3}}{3}\right)x^3 \\ &= x^3\left(\ln x - 1\right) + \frac{1}{6}. \end{aligned}$$

and a general solution is given by

$$y(x) = c_1 x^2 + c_2 x^3 + x^3\left(\ln x - 1\right) + \frac{1}{6}.$$

Chapter 5

Higher-Order Linear Differential Equations

We now show that our discussion in the previous chapter of second-order linear equations generalizes in a very natural way to the general nth-order linear differential equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x), \qquad (5.1)$$

where $a_0(x), a_1(x), ..., a_n(x)$ and $b(x)$ depend only on x, not y. When $a_0, a_1, ..., a_n$ are all constants, we say equation (5.1) has constant coefficients; otherwise it has variable coefficients. If $b(x) = 0$, equation (5.1) is called homogeneous; otherwise it is nonhomogeneous.

In developing a basic theory, we assume that $a_0(x), a_1(x), ..., a_n(x)$ and $g(x)$ are all continuous on an interval I and $a_n(x) \neq 0$ on I. Then, on dividing by $a_n(x)$, we can rewrite (5.1) in the standard form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x),$$

where the functions $p_0(x), p_1(x), ..., p_{n-1}(x)$ and $g(x)$ are continuous on I.

For a linear higher-order differential equation, the initial value problem always has a unique solution.

Theorem 5.1 *Suppose $p_0(x), p_1(x), ..., p_{n-1}(x)$ and $g(x)$ are each continuous on an open interval* 1 *that contains the point x_0. Then, for any choice of the initial values $\gamma_0, \gamma_1, ..., \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval I to the initial value problem*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x),$$

$$y(x_0) = \gamma_0, \quad y'(x_0) = \gamma_1, \quad ..., \quad y^{(n-1)}(x_0) = \gamma_{n-1}.$$

Just as in the second-order case, a homogeneous nth-order linear differential equation has the valuable property that any superposition, or linear combination

$$c_1y_1 + c_2y_2 + ... + c_ny_n$$

of solutions of the equation is again a solution.

Definition 5.1 The m functions $f_1, f_2, ..., f_m$ are said to be linearly dependent on an interval I if at least one of them can be expressed as a linear combination of the others on I; equivalently, they are linearly dependent if there exist constants $c_1, c_2, ..., c_m$, not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + ... + c_mf_m(x) = 0$$

for all x in I. Otherwise, they are said to be linearly independent on I.

Imagine now that we have found n solutions $y_1, ..., y_n$ to the nth-order homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{5.2}$$

Is it true that every solution to (5.2) can be represented by

$$c_1y_1 + c_2y_2 + ... + c_ny_n$$

for appropriate choices of the constants $c_1, ..., c_n$? The answer is yes, provided the solutions $y_1, ..., y_n$ satisfy a certain property that we now derive.

Theorem 5.2 *Let $y_1, ..., y_n$ be n solutions on an open interval I of*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{5.3}$$

where $p_0(x), p_1(x), ..., p_{n-1}(x)$ are continuous on 1. If these solutions are linearly independent on I, then every solution of (5.3) on I can be expressed in the form

$$c_1y_1 + c_2y_2 + ... + c_ny_n$$

where $c_1, ..., c_n$ are constants.

Thus every solution of a homogeneous nth-order linear differential equation is a linear combination

$$c_1y_1 + c_2y_2 + ... + c_ny_n$$

of any n given linearly independent solutions. On this basis we call such a linear combination a **general solution** of the differential equation.

We now consider the nonhomogeneous nth-order linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x) \tag{5.4}$$

with associated homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{5.5}$$

If we combine the superposition principle with the representation theorem for solutions of the homogeneous equation, we obtain the following representation theorem for nonhomogeneous equations.

Theorem 5.3 *Let $y_p(x)$ be a particular solution of the nonhomogeneous equation (5.4) on an open interval I where the functions $p_0(x), p_1(x), ..., p_{n-1}(x)$ and $g(x)$ are continuous. Let $y_1, ..., y_n$ be linearly independent solutions of the associated homogeneous equation in (5.5). Then every solution of (5.4) on the interval I can be expressed in the form*

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) + y_p(x). \end{aligned}$$

5.1 Homogeneous Equations with Constant Coefficients

Consider the nth-order linear equation with constant coefficients

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0, \tag{5.6}$$

where $a_0, a_1, ..., a_n$ are all real constants.

In Section 4.3 we substituted $y(x) = e^{rx}$ in the second-order equation

$$ay'' + by' + cy + 0$$

to derive the auxiliary equation that r must satisfy. To carry out this technique in the general case, we substitute $y(x) = e^{rx}$ in Eq. (5.6), and with the aid of

$$\frac{d^k}{dx^k}e^{rx} = r^ke^{rx}$$

we find the result to be

$$a_nr^ne^{rx} + a_{n-1}r^{n-1}e^{rx} + \cdots + a_1re^{rx} + a_0e^{rx} = 0,$$

that is,

$$e^{rx}\left(a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0\right) = 0.$$

Because e^{rx} is never zero, we see that $y = e^{rx}$ will be a solution of Eq. (5.6) precisely when r is a root of the equation

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{5.7}$$

This equation is called the auxiliary equation or characteristic equation of the differential equation in (5.6). Our problem, then, is reduced to the solution of this purely algebraic equation.

According to the fundamental theorem of algebra, every nth-degree polynomial—such as the one on the left-hand side of Eq. (5.7)—has n zeros, though not necessarily distinct and not necessarily real. Hence we can write the auxiliary equation in the form

$$(r - r_1)(r - r_2)\cdots(r - r_n) = 0,$$

with n roots, say $r_1, r_2, ..., r_n$. Finding the exact values of these zeros may be difficult or even impossible; the quadratic formula is sufficient for second-degree equations, but for equations of higher degree we may need either to use a factorization or to apply a numerical technique such as Newton's method or to use a calculator/computer solver.

Distinct Real Roots: If the roots $r_1, r_2, ..., r_n$ of the auxiliary equation in (5.7) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x}$$

is a general solution of Eq. (5.6).

Repeated Real Roots: If the auxiliary equation in (5.7) has a repeated root r of multiplicity k, then the part of a general solution of the differential equation in (5.6) corresponding to r is of the form

$$c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx} ... + c_k x^{k-1} e^{rx}.$$

Complex Roots: If the auxiliary equation has complex roots, they must occur in conjugate pairs, $\alpha \pm \beta i$, since the coefficients $a_0, a_1, ..., a_n$ are real numbers. Provided that none of the roots is repeated, the general solution of Eq. (5.6) is still of the form

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x}.$$

However, just as for the second order equation (Section 4.3), we can replace the complex-valued solutions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ by the real-valued solutions

$$e^{\alpha x} \cos(\beta x), \qquad e^{\alpha x} \sin(\beta x).$$

Repeated Complex Roots: If a complex root $\alpha + \beta i$ is repeated k times, the complex conjugate $\alpha - \beta i$ is also repeated k times. Corresponding to these $2k$ complex-valued solutions, we can find $2k$ real-valued solutions by noting that the real and imaginary parts of $e^{(\alpha+\beta i)x}, xe^{(\alpha+\beta i)x}, ..., x^{k-1}e^{(\alpha+\beta i)x}$ are also linearly independent solutions:

$$\begin{aligned}
&e^{\alpha x} \cos(\beta x), \qquad e^{\alpha x} \sin(\beta x), \\
&xe^{\alpha x} \cos(\beta x), \qquad xe^{\alpha x} \sin(\beta x), \\
&\vdots \\
&x^{k-1}e^{\alpha x} \cos(\beta x), \qquad x^{k-1}e^{\alpha x} \sin(\beta x).
\end{aligned}$$

Example 5.1 Find the general solution of

$$y^{(3)} + 3y'' - 10y' = 0. \tag{5.8}$$

Solution: Assuming that $y = e^{rx}$, we must determine r by solving the auxiliary equation

$$r^3 + 3r^2 - 10r = 0.$$

We solve by factoring:

$$r^3 + 3r^2 - 10r = r\left(r^2 + 3r - 10\right) = r(r+5)(r-2) = 0,$$

and so the auxiliary equation has the three distinct real roots $r_1 = 0, r_2 = -5, r_3 = 2$. Therefore the general solution of Eq. (5.8) is

$$y(x) = c_1 + c_2 e^{-5x} + c_3 e^{2x}.$$

Example 5.2 Find the general solution of

$$y^{(4)} + 2y'' + y = 0. \tag{5.9}$$

Solution: The auxiliary equation is

$$r^4 + 2r^2 + 1 = \left(r^2 + 1\right)^2 = 0,$$

which has the repeated complex roots $r = i, -i, i, -i$.Therefore the general solution of Eq. (5.9) is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$

Example 5.3 Find the general solution of

$$y^{(4)} + 4y = 0. \tag{5.10}$$

Solution: The auxiliary equation is

$$r^4 + 4 = \left(r^2\right)^2 - (2i)^2 = \left(r^2 - 2i\right)\left(r^2 + 2i\right) = 0$$

and its four roots are $\pm\sqrt{\pm 2i}$. Since $i = e^{i\pi/2}$ and $-i = e^{i3\pi/2}$, we find that

$$\sqrt{2i} = \left(2e^{i\pi/2}\right)^{1/2} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i$$
$$\sqrt{-2i} = \left(2e^{i3\pi/2}\right)^{1/2} = \sqrt{2}e^{i3\pi/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1 + i.$$

Thus the four roots of the auxiliary equation are $1 \pm i, -1 \pm i$. These two pairs of complex conjugate roots give the general solution of Eq. (5.10)

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x + c_3 e^{-x} \cos x + c_4 e^{-x} \sin x.$$

Example 5.4 Find the general solution of the fifth-order differential equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0.$$

Solution: The auxiliary equation is

$$9r^5 - 6r^4 + r^3 = r^3\left(9r^2 - 6r^2 + 1\right) = r^3\left(3r - 1\right)^2 = 0.$$

It has the triple root $r = 0$ and the double root $r = 1/3$. The triple root $r = 0$ contributes

$$c_1 + c_2x + c_3x^2$$

to the solution, while the double root $r = 1/3$ contributes

$$c_4e^{x/3} + c_5xe^{x/3}.$$

Hence the general solution of the given differential equation is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4e^{x/3} + c_5xe^{x/3}.$$

Example 5.5 Find the general solution of

$$y^{(3)} + y' - 10y = 0.$$

Solution: The auxiliary equation is the cubic equation

$$r^3 + r - 10 = 0.$$

By a standard theorem of elementary algebra, the only possible rational roots are the divisors of the constant term 10, namely $\pm1, \pm2, \pm5, \pm10$. By trial and error (if not by inspection) we discover the root $r = 2$ and so $r-2$ is a factor of $r^3 + r - 10$. Thus, using polynomial division, we get

$$r^3 + r - 10 = (r-2)(r^2 + 2r + 5)$$

The roots of the auxiliary equation are $2, -1 \pm 2i$. The three roots we have found now yield the general solution

$$y(x) = c_1e^{2x} + c_2e^{-x}\cos 2x + c_3e^{-x}\sin 2x.$$

Example 5.6 The roots of the characteristic equation of a certain differential equation are $0, 5, 5, 7, 7, 7, 7, 2 \pm 3i, 2 \pm 3i$. Write the general solution of this homogeneous differential equation.

Solution: The solution can be read directly from the list of roots. It is

$$y(x) = c_1 + c_2e^{5x} + c_3xe^{5x} + c_4e^{7x} + c_5xe^{7x} + c_6x^2e^{7x} + c_7x^3e^{7x} + c_8e^{2x}\cos 3x + c_9e^{2x}\sin 3x + c_{10}xe^{2x}\cos 3x + c_{11}xe^{2x}\sin 3x.$$

5.2 Undetermined Coefficients for Higher Order Equation

In this section we consider the nonhomogeneous constant coefficient equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \qquad (5.11)$$

where $a_0, a_1, ..., a_n$ are all real constants and $g(x)$ is a sum of polynomials, exponentials, sines, and cosines, or products of such functions.

The general solution of (5.11) is $y(x) = y_h(x) + y_p(x)$, where y_p is a particular solution of (5.11) and y_h is the general solution of the corresponding homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The procedure that we use to find y_p is a generalization of the method that we used with nonhomogeneous second order equations with constant coefficients in Section 4.4, and is again called method of undetermined coefficients. Since the underlying ideas are the same as those in Section 4.4, we'll give an informal presentation based on examples.

Example 5.7 Determine a suitable form for $y_p(x)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

(a) $y''' - 3y'' + 3y' - y = 4e^x$

The auxiliary equation for the homogeneous equation corresponding to this equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0,$$

so the general solution of the corresponding homogeneous equation is

$$y_h(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

To find a particular solution $y_p(x)$, we start by assuming that $y_p(x) = Ae^x$. However, since e^x, xe^x, x^2e^x are all solutions of the homogeneous equation, we must multiply this initial choice by x^3. Thus our final assumption is that

$$y_p(x) = Ax^3 e^x.$$

(b) $y^{(4)} + 2y'' + y = 2\sin x - 5\cos x$

First we solve the homogeneous equation

$$y^{(4)} + 2y'' + y = 0.$$

The auxiliary equation is

$$r^4 + 2r^2 + 1 = \left(r^2 + 1\right)^2 = 0,$$

and the roots are $i, -i, i, -i$; hence

$$y_h(x) = c_1 \sin x + c_2 \cos x + c_3 x \sin x + c_4 x \cos x.$$

Our initial assumption for a particular solution is $y_p(x) = A \sin x + B \cos x$, but we must multiply this choice by x^2 to make it different from all solutions of the homogeneous equation. Thus our final assumption is

$$y_p(x) = Ax^2 \sin x + Bx^2 \cos x.$$

(c) $y^{(3)} + 9y' = x \sin 3x + x^2 e^{2x}$

The homogeneous equation corresponding to this equation is $y^{(3)} + 9y' = 0$, with the auxiliary equation

$$r^3 + 9r = r\left(r^2 + 9\right) = 0.$$

The roots of the auxiliary equation are $0, \pm 3i$; hence

$$y_h(x) = c_1 + c_2 \sin 3x + c_3 \cos 3x.$$

We can write a particular solution as the sum of particular solutions of the differential equations

$$y^{(3)} + 9y' = x \sin 3x, \qquad y^{(3)} + 9y' = x^2 e^{2x}.$$

Our initial choice for a particular solution $y_{p_1}(x)$ of the first equation is

$$y_{p_1}(x) = (A_1 x + A_2) \sin 3x + (A_3 x + A_4) \cos 3x,$$

but since $\sin 3x$ is a solution of the homogeneous equation, we multiply by x. Thus

$$\begin{aligned} y_{p_1}(x) &= x\left[(A_1 x + A_2) \sin 3x + (A_3 x + A_4) \cos 3x\right] \\ &= \left(A_1 x^2 + A_2 x\right) \sin 3x + \left(A_3 x^2 + A_4 x\right) \cos 3x. \end{aligned}$$

For the second equation we choose

$$y_{p_2}(x) = e^{2x}\left(A_5 x^2 + A_6 x + A_7\right),$$

and there is no need to modify this initial choice since e^{2x}, xe^{2x} and $x^2 e^{2x}$ are not solutions of the homogeneous equation. Hence a particular solution of the given equation is

$$y_p(x) = \left(A_1x^2 + A_2x\right)\sin 3x + \left(A_3x^2 + A_4x\right)\cos 3x + e^{2x}\left(A_5x^2 + A_6x + A_7\right).$$

Example 5.8 Find the general solution of

$$y^{(3)} + y'' = 3e^x + 4x^2. \tag{5.12}$$

Solution: The auxiliary equation

$$r^3 + r^2 = r^2(r+1) = 0$$

has roots $r_1 = r_2 = 0$ and $r^3 = -1$, so the complementary function is

$$y_h(x) = c_1 + c_2x + c_3e^{-x}.$$

As a first step toward our particular solution, we choose

$$y_p(x) = A_1e^x + A_2x^2 + A_3x + A_4.$$

The part A_1e^x corresponding to $3e^x$ does not duplicate any part of $y_h(x)$, but the part $A_2x^2 + A_3x + A_4$ must be multiplied by x^2 to eliminate duplication. Hence we take

$$y_p(x) = A_1e^x + A_2x^4 + A_3x^3 + A_4x^2.$$

Next, we differentiate $y_p(x)$ three times

$$\begin{aligned} y_p'(x) &= A_1e^x + 4A_2x^3 + 3A_3x^2 + 2A_4x, \\ y_p''(x) &= A_1e^x + 12A_2x^2 + 6A_3x + 2A_4, \\ y_p^{(3)}(x) &= A_1e^x + 24A_2x + 6A_3. \end{aligned}$$

Substitution of these derivatives in Eq. (5.12) yields

$$2A_1e^x + (2A_4 + 6A_3) + (6A_3 + 24A_2)\,x + 12A_2x^2 = 3e^x + 4x^2.$$

Upon equating coefficients of like terms, we get

$$2A_1 = 3, \quad 2A_4 + 6A_3 = 0, \quad 6A_3 + 24A_2 = 0, \quad 12A_2 = 4$$

and so

$$A_1 = \frac{3}{2}, \quad A_2 = \frac{1}{3}, \quad A_3 = -\frac{4}{3}, \quad A_4 = 4.$$

Hence the desired particular solution is

$$y_p(x) = \frac{3}{2}e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2.$$

The general solution of Eq. (5.12) is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 + c_2 x + c_3 e^{-x} + \frac{3}{2}e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2. \end{aligned}$$

Chapter 6

Laplace Transform Methods

The method of Laplace transforms is one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.

6.1 Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, b]$ for every $b > a$, then the improper integral of g over $[0, \infty)$ is defined as

$$\int_a^\infty g(x)dx = \lim_{b\to\infty} \int_a^b g(x)dx.$$

If the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Definition 6.1 Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follows:

$$F(s) = \int_0^\infty e^{-st} f(t)dt, \tag{6.1}$$

for all values of s for which the improper integral converges

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f(t)$ into the function $F(s)$,

$$\mathcal{L} : f(t) \to F(s)$$

Thus, (6.1) can be expressed as

$$F(s) = \mathcal{L}\left\{f(t)\right\}.$$

The Laplace transform $\mathcal{L}\left\{f(t)\right\}$ exists if f satisfies certain conditions, such as those stated in the following theorem.

A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity (the one-sided limits exist as finite numbers). A function $f(t)$ is said to be piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

For example, from the graph of the function $f(t)$,

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ (t-2)^2, & 2 \le t \le 3, \end{cases}$$

we see that $f(t)$ is continuous on the intervals $(0, 1)$, $(1, 2)$, and $(2, 3]$. Moreover, at the points of discontinuity, $t = 0, 1, 2$, the function has jump discontinuities, since the one-sided limits exist as finite numbers. In particular, at $t = 1$, the left-hand limit is 1 and the right-hand limit is 2. Therefore $f(t)$ is piecewise continuous on $[0, 3]$.

In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin,

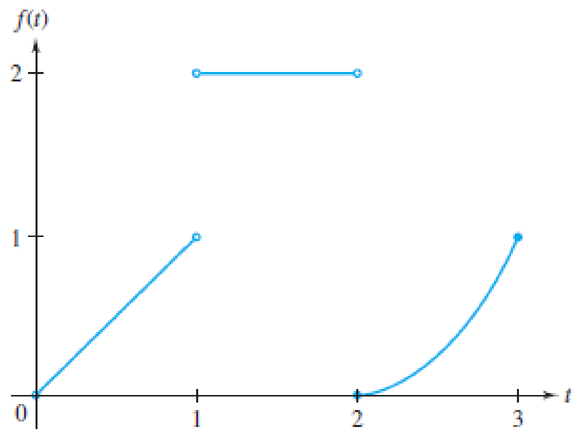
$$\lim_{t\to 0^+} \frac{1}{t} = +\infty, \qquad \lim_{t\to 0^-} \frac{1}{t} = -\infty.$$

Theorem 6.1 *Suppose that*

(1) f is piecewise continuous on the interval $0 \le t \le A$ for any positive A.
(2) $|f(t)| \le Ke^{at}$ when $t \ge M$. Here K, a, and M are real constants, K and M necessarily positive.

Then the Laplace transform $F(s) = \mathcal{L}\left\{f(t)\right\}$, defined by (6.1), exists for $s > a$.

Example 6.1 Determine the Laplace transform of the constant function $f(t) = 1$.



Solution: The definition of the Laplace transform in (6.1) gives

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st}\cdot 1dt\\ &= \lim_{b\to\infty}\int_0^b e^{-st}dt\\ &= \lim_{b\to\infty}\left.\frac{e^{-st}}{-s}\right]_0^b\\ &= \lim_{b\to\infty}\left(\frac{e^{-bs}}{-s}+1\right)\end{aligned}$$

Since $\lim_{b\to\infty} e^{-bs}\to 0$ when $s>0$, we get

$$F(s)=\frac{1}{s}\quad \text{for} \quad s>0.$$

When $s\le 0$, the integral $\int_0^\infty e^{-st}dt$ diverges. Hence $F(s)=\frac{1}{s}$, with the domain of $F(s)$ being all $s>0$.

Example 6.2 Find the Laplace transform of $f(t)=t$.
Solution: Using the definition of the transform with $f(t)=t$,

$$F(s)=\int_0^\infty te^{-st}dt \tag{6.2}$$

If $s \neq 0$, integrating by parts yields

$$\begin{aligned}
F(s) &= \lim_{b\to\infty} \int_0^b te^{-st}dt \\
&= \lim_{b\to\infty} \left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\Big]_0^b \right) \\
&= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases}
\end{aligned}$$

If $s = 0$, the integral in (6.2) becomes

$$\lim_{b\to\infty} \int_0^b tdt = \lim_{b\to\infty} t^2\Big]_0^b = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \qquad s > 0.$$

Example 6.3 Determine the Laplace transform of the finction

$$f(t) = \begin{cases} e^{2t}, & 0 < t < 3, \\ 4, & t > 3. \end{cases}$$

Solution: Notice that the function f has jump discontinuities at $t = 3$. Splitting the integral in the definition of Laplace transform, we get

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^3 e^{-st}e^{2t}dt + \int_3^\infty e^{-st} \cdot 4dt \\
&= \int_0^3 e^{-(s-2)t}dt + 4 \lim_{b\to\infty} \int_3^b e^{-st}dt \\
&= \frac{e^{-(s-2)t}}{-(s-2)}\Big]_0^3 + 4 \lim_{b\to\infty} \frac{e^{-st}}{-s}\Big]_3^b \\
&= \left(\frac{e^{-3(s-2)}}{-(s-2)} + \frac{1}{s-2} \right) + 4 \lim_{b\to\infty} \left(\frac{e^{-sb}}{-s} + \frac{e^{-3s}}{s} \right) \\
&= \frac{1 - e^{-3(s-2)}}{s-2} + 4\frac{e^{-3s}}{s} \qquad \text{for} \quad s > 0.
\end{aligned}$$

The given table lists the Laplace transforms of some of the elementary functions. You should become familiar with these, since they are frequently encountered in solving linear differential equations with constant coefficients. The entries in the table can be derived from the definition of the Laplace transform.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

6.2 Properties of the Laplace transform

An important property of the Laplace transform is its linearity. That is, the Laplace transform is a linear operator. This property results directly from the linearity properties of integration.

Theorem 6.2 *(**Linearity of the Transform**) If a and b are constants, then*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for all s such that the Laplace transforms of the functions f and g both exist.

Example 6.4 Determine $\mathcal{L}\left\{\sin 3t - 2e^{7t} + 5\right\}$.

Solution: From the linearity property, we obtain

$$\mathcal{L}\left\{\sin 3t - 2e^{4t} + 5\right\} = \mathcal{L}\{\sin 3t\} - 2\mathcal{L}\left\{e^{4t}\right\} + 5\mathcal{L}\{1\}.$$

From the table,

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}, \qquad s > 0,$$
$$\mathcal{L}\left\{e^{4t}\right\} = \frac{1}{s-4}, \qquad s > 4,$$
$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

Using these results, we find

$$\begin{aligned}\mathcal{L}\left\{\sin 3t - 2e^{7t} + 5\right\} &= \frac{3}{s^2+9} - 2\left(\frac{1}{s-7}\right) + 5\left(\frac{1}{s}\right)\\ &= \frac{3}{s^2+9} - \frac{2}{s-7} + \frac{5}{s}, \qquad s > 4.\end{aligned}$$

Example 6.5 Determine $\mathcal{L}\{\sinh 2t\}$.

Solution: Recall that

$$\sinh 2t = \frac{e^{2t} - e^{-2t}}{2}.$$

Therefore, by linearity,

$$\begin{aligned}\mathcal{L}\{\sinh 2t\} &= \mathcal{L}\left\{\frac{e^{2t} - e^{-2t}}{2}\right\}\\ &= \frac{1}{2}\mathcal{L}\left\{e^{2t} - e^{-2t}\right\}\\ &= \frac{1}{2}\left(\mathcal{L}\left\{e^{2t}\right\} - \mathcal{L}\left\{e^{-2t}\right\}\right)\\ &= \frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s+2}\right)\\ &= \frac{1}{2}\frac{4}{s^2-4}\\ &= \frac{2}{s^2-4} \qquad \text{for } \ s > 2.\end{aligned}$$

Theorem 6.3 ***(Shifting Property)*** *If the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > \alpha$, then*

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a) \tag{6.3}$$

for $s > a + \alpha$.

Example 6.6 Determine $\mathcal{L}\left\{e^{3t}\cos 4t\right\}$.

Solution: From the table

$$\mathcal{L}\{\cos 4t\} = F(s) = \frac{s}{s^2+16}.$$

Thus, by the shifting property, we have

$$\mathcal{L}\left\{e^{3t}\cos 4t\right\} = F(s-3) = \frac{s-3}{(s-3)^2+16}.$$

Example 6.7 Determine $\mathcal{L}\left\{t^2 e^{-4t}\right\}$.

Solution: From the table

$$\mathcal{L}\left\{t^2\right\} = F(s) = \frac{2}{s^3}.$$

Thus, by the shifting property, we have

$$\mathcal{L}\left\{t^2 e^{-4t}\right\} = F(s+4) = \frac{2}{(s+4)^3}.$$

Theorem 6.4 ***(Multiplication by*** t^n***)*** *Let $F(s) = \mathcal{L}\{f(t)\}$ and assume $f(t)$ is piecewise continuous for $t \geq 0$, and of exponential order a. Then, for $s > a$,*

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s). \tag{6.4}$$

Example 6.8 Determine $\mathcal{L}\{t\sin 3t\}$.

Solution: We already know that

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}.$$

Hence, using formula (6.4) with $n=1$, we have

$$\begin{aligned}\mathcal{L}\{t\cos 3t\} &= (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)\\ &= (-1)(3)\frac{d}{ds}\left(\frac{1}{s^2+9}\right)\\ &= (-3)\left(\frac{-2s}{(s^2+9)^2}\right)\\ &= \frac{6s}{(s^2+9)^2}.\end{aligned}$$

Example 6.9 Determine $\mathcal{L}\left\{t\sin^2 t\right\}$.

Solution: Recall that

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t).$$

By the linearity property we obtain

$$\begin{aligned}\mathcal{L}\left\{\sin^2 t\right\} &= \mathcal{L}\left\{\frac{1}{2}(1 - \cos 2t)\right\} \\ &= \frac{1}{2}(\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}) \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).\end{aligned}$$

Hence, using formula (6.4) with $n = 1$ and $f(t) = \sin^2 t$, we have

$$\begin{aligned}\mathcal{L}\left\{t\sin^2 t\right\} &= (-1)^1 \frac{d}{ds}\left(\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right) \\ &= -\frac{1}{2}\left(\frac{-1}{s^2} - \frac{\left(s^2+4\right)(1) - (s)(2s)}{\left(s^2+4\right)^2}\right) \\ &= -\frac{1}{2}\left(\frac{-1}{s^2} - \frac{4-s^2}{\left(s^2+4\right)^2}\right) \\ &= \frac{1}{2}\left(\frac{1}{s^2} + \frac{4-s^2}{\left(s^2+4\right)^2}\right).\end{aligned}$$

Example 6.10 Determine $\mathcal{L}\left\{te^{2t}\cos 5t\right\}$.

Solution: We already know that

$$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+25}.$$

Using the shifting property (6.4), we have

$$\mathcal{L}\left\{e^{2t}\cos 3t\right\} = \frac{s-2}{(s-2)^2+25}.$$

Hence, using formula (6.4) with $n = 1$ and $f(t) = e^{2t}\cos 3t$, we get

$$\begin{aligned}\mathcal{L}\left\{te^{2t}\cos 5t\right\} &= (-1)\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+25}\right)\\ &= -\left(\frac{\left[(s-2)^2+25\right]\cdot 1-(s-2)\cdot 2(s-2)}{\left[(s-2)^2+25\right]^2}\right)\\ &= \frac{(s-2)^2-25}{\left[(s-2)^2+25\right]^2}.\end{aligned}$$

Theorem 6.5 ***(Laplace Transform of Higher-Order Derivatives)*** *Let $f(t), f'(t), ..., f^{(n-1)}(t)$ be continuous for $t \geq 0$, and let $f^{(n)}(t)$ be piecewise continuous for $t \geq 0$, with all these functions of exponential order α. Then, for $s > \alpha$,*

$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f\}(s)$.

For example,

$$\begin{aligned}\mathcal{L}\{f'\}(s) &= sF(s) - f(0),\\ \mathcal{L}\{f''\}(s) &= s^2F(s) - f(0)s - f'(0),\\ \mathcal{L}\{f'''\}(s) &= s^3F(s) - f(0)s^2 - f'(0)s - f''(0).\end{aligned}$$

The last theorem tells us that by using the Laplace transform we can replace "differentiation with respect to t" with "multiplication by s," thereby converting a differential equation into an algebraic one. This idea is explored in Section 7.5. For now, we show how Theorem 4 can be helpful in computing a Laplace transform.

Example 6.11 The transfer function of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, when all initial conditions are zero. If a linear system is governed by the differential equation

$$y''(t) + 6y'(t) + 10y(t) = g(t), \qquad t > 0.$$

Determine the transfer function $H(s) = Y(s)/G(s)$ for this system.

Solution: Let $F(s) = \mathcal{L}\{y\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Taking the Laplace transform of both sides of the differential equation gives

$$\mathcal{L}\{y'' + 6y' + 10y\} = \mathcal{L}\{g\}.$$

We can use the linearity property of the Laplace transform and the theorem on the Laplace transform of higher-order derivatives to obtain

$$\begin{aligned} \mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} &= G(s) \\ s^2Y(s) - y(0)s - y'(0) + 6\,(sY(s) - y(0)) + 10Y(s) &= G(s). \end{aligned}$$

Since all initial conditions are zero, we have

$$\begin{aligned} s^2Y(s) + 6sY(s) + 10Y(s) &= G(s), \\ \left(s^2 + 6s + 10\right)Y(s) &= G(s). \end{aligned}$$

We can now express the transfer function $H(s) = Y(s)/G(s)$ by

$$\frac{Y(s)}{G(s)} = \frac{1}{(s^2 + 6s + 10)}.$$

Theorem 6.6 *If the function f is piecewise continuous for $t \geq 0$ and is of exponential order α, then*

$$\lim_{s\to\infty} F(s) = 0. \tag{6.5}$$

The condition in (6.5) severely limits the functions that can be Laplace transforms. For example, the function $F(s) = s/(s+1)$ cannot be the Laplace transform of any function because its limit as $s \to \infty$ is 1, not 0. More generally, a rational function (a quotient of two polynomials) can be a Laplace transform only if the degree of its numerator is less than that of its denominator.

6.3 Inverse Laplace Transform

We start this section with the following result.

Theorem 6.7 *Suppose that the functions $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and are of exponential order as $t \to +\infty$, so that their Laplace transforms $F(s)$ and $G(s)$ both exist. If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.*

Thus two piecewise continuous functions of exponential order with the same Laplace transform can differ only at their isolated points of discontinuity. This is of no importance in most practical applications, so we may regard inverse Laplace transforms as being essentially unique. Morwover, if $F(s)$ is the transform of some continuous function $f(t)$, then $f(t)$ is uniquely determined. This observation allows us to make the following definition:

Definition 6.2 Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\left\{f\right\} = F(s),$$

then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and employ the notation

$$\mathcal{L}^{-1}\left\{F\right\} = f.$$

Usually the Laplace transform tables will be a great help in determining the inverse Laplace transform of a given function $F(s)$.

The linearity property of the inverse Laplace transform is inherited from the linearity of the operator $\mathcal{L}$.

Theorem 6.8 *Assume that $\mathcal{L}^{-1}\left\{F\right\}$ and $\mathcal{L}^{-1}\left\{G\right\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then*

$$\mathcal{L}^{-1}\left\{F + G\right\} = \mathcal{L}^{-1}\left\{F\right\} + \mathcal{L}^{-1}\left\{G\right\},$$
$$\mathcal{L}^{-1}\left\{cF\right\} = c\mathcal{L}^{-1}\left\{F\right\}.$$

Example 6.12 Determine the inverse Laplace transform of the given function.

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

Recall that $\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$. Using the linearity property we get

$$\frac{1}{2}\mathcal{L}\left\{t^2\right\} = \frac{1}{s^3},$$
$$\mathcal{L}\left\{\frac{1}{2}t^2\right\} = \frac{1}{s^3}.$$

Thus $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$.

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

We know that $\mathcal{L}\left\{e^{-2t}\right\} = \frac{1}{s+2}$. So, $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$.

(c) $\mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\}$

Referring to the Laplace transform tables, we see that $\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}$. We now use the linearity property

$$\frac{1}{4}\mathcal{L}\{\sin 4t\} = \frac{1}{s^2+16},$$
$$\frac{3}{4}\mathcal{L}\{\sin 4t\} = \frac{3}{s^2+16},$$
$$\mathcal{L}\left\{\frac{3}{4}\sin 4t\right\} = \frac{3}{s^2+16}.$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\} = \frac{3}{4}\sin 4t.$$

(d) $\mathcal{L}^{-1}\left\{\frac{5}{(s-7)^4}\right\}$

Referring to the Laplace transform tables, we find that $\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4}$. The shifting property now gives

$$\mathcal{L}\left\{e^{7t}t^3\right\} = \frac{3!}{(s-7)^4}.$$

Using the linearity property, we find

$$\frac{1}{6}\mathcal{L}\left\{e^{7t}t^3\right\} = \frac{1}{(s-7)^4},$$
$$\frac{5}{6}\mathcal{L}\left\{e^{7t}t^3\right\} = \frac{5}{(s-7)^4},$$
$$\mathcal{L}\left\{\frac{5}{6}e^{7t}t^3\right\} = \frac{5}{(s-7)^4}.$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{5}{(s-7)^4}\right\} = \frac{5}{6}e^{7t}t^3.$$

(d) $\mathcal{L}^{-1}\left\{\frac{s-5}{s^2-2s+5}\right\}$

By completing the square, the quadratic in the denominator can be written as

$$s^2 - 2s + 5 = s^2 - 2s + 1 - 1 + 5 = (s-1)^2 + 4.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{s-5}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-1)-4}{(s-1)^2+4}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}.$$

This suggests that we use the formulas

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4},$$
$$\mathcal{L}\left\{e^t\cos 2t\right\} = \frac{s-1}{(s-1)^2+4}$$

and

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4},$$
$$\mathcal{L}\left\{e^t\sin 2t\right\} = \frac{2}{(s-1)^2+4}.$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\} = e^t\cos 2t, \qquad \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\} = e^t\sin 2t.$$

Finally, we can determine the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{s-5}{s^2-2s+5}\right\} = e^t\cos 2t - 2e^t\sin 2t.$$

Example 6.13 Determine $\mathcal{L}^{-1}\left\{\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}\right\}$.

Solution: We begin by finding the partial fraction expansion for $\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}$. The denominator consists of three distinct linear factors, so the expansion has the form

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+5}, \tag{6.6}$$

where A, B, and C are real numbers to be determined. We multiply (6.6) by $(s-1)(s+2)(s+5)$ and find

$$s^2-26s-47 = A(s+2)(s+5) + B(s-1)(s+5) + C(s-1)(s+2). \tag{6.7}$$

for all s. Setting $s=1$ in Eq. (6.7) gives

$$-72 = 18A \implies A = -4.$$

Next, setting $s = -2$ gives

$$9 = -9B \implies B = -1.$$

Finally, letting $s = -5$, we similarly find that

$$108 = 18C \implies C = 6.$$

Now that we have obtained the partial fraction expansion (6.6), we use linearity to compute

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}\right\} &= \mathcal{L}^{-1}\left\{\frac{-4}{s-1}+\frac{-1}{s+2}+\frac{6}{s+5}\right\}\\ &= -4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+6\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}\\ &= -4e^{t}-e^{-2t}+6e^{-5t}.\end{aligned}$$

Example 6.14 Determine $\mathcal{L}^{-1}\left\{\frac{s^2+8s+6}{(s+1)(s^2+4)}\right\}$.

Solution: We first observe that the quadratic factor (s^2+4) is irreducible. The partial fraction expansion has the form

$$\frac{s^2+8s+6}{(s+1)(s^2+4)} = \frac{A}{s+1}+\frac{Cs+D}{s^2+4}. \tag{6.8}$$

When we multiply both sides by $(s+1)(s^2+4)$, we obtain

$$s^2+8s+6 = A(s^2+4)+(Cs+D)(s+1), \tag{6.9}$$

for all s. In equation (6.9), let's put $s = -1, 0$ and 1. We find

$$\begin{aligned}&s=-1 \implies -1=5A \implies A=\frac{-1}{5}.\\ &s=0 \implies 6=4A+D \implies D=\frac{34}{5},\\ &s=1 \implies 15=5A+2C+2D \implies C=\frac{6}{5}.\end{aligned}$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2+8s+6}{(s+1)(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}}{s+1}+\frac{\frac{6}{5}s+\frac{34}{5}}{s^2+4}\right\}\\ &= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{6}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{17}{5}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\\ &= -\frac{1}{5}e^{-t}+\frac{6}{5}\cos 2t+\frac{17}{5}\sin 2t.\end{aligned}$$

The multiplication rule by t^n,

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n F^{(n)}(s),$$

can be expressed in terms of the inverse Laplace transform as

$$\mathcal{L}^{-1}\left\{F^{(n)}(s)\right\} = (-t)^n f(t), \qquad (6.10)$$

where $F(s) = \mathcal{L}\left\{f(t)\right\}$.

Example 6.15 Determine $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

Solution: We note first that

$$\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = \frac{-2s}{(s^2+4)^2}.$$

By formula (6.10) with $n = 1$ and $F(s) = \frac{1}{s^2+4}$, it follws that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+4)^2}\right\} &= \mathcal{L}^{-1}\left\{F'(s)\right\}\\ -2\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} &= (-t)^1\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\\ -2\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} &= (-t)\frac{1}{2}\sin 2t.\end{aligned}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\sin 2t.$$

Example 6.16 Compute

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s-4}{s-3}\right)\right\}.$$

Solution: We are looking for $f(t) = \mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \ln\left(\frac{s-4}{s-3}\right)$. According to formula (6.10) with $n = 1$,

$$\mathcal{L}^{-1}\{F'(s)\} = (-t)f(t) \quad \Longrightarrow \quad f(t) = \frac{-1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

Since

$$F(s) = \ln\left(\frac{s-4}{s-3}\right) = \ln(s-4) - \ln(s-3),$$

we have

$$F'(s) = \frac{1}{s-4} - \frac{1}{s-3},$$

and consequently,

$$\begin{aligned} \mathcal{L}^{-1}\{F'(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-4} - \frac{1}{s-3}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} \\ &= e^{4t} - e^{3t}. \end{aligned}$$

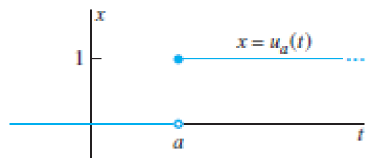
Thus,

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= f(t) \\ &= -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} \\ &= -\frac{1}{t}\left(e^{4t} - e^{3t}\right) \\ &= \frac{e^{3t} - e^{4t}}{t}. \end{aligned}$$

6.4 Transforms of Discontinuous Functions

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the unit step function, or Heaviside



function. This function will be denoted by u_a, $a \geq 0$, and is defined by

$$u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases} \tag{6.11}$$

The function u_a has its jump at $t = a$. By multiplying by a constant M, the height of the jump can also be modified:

$$Mu_a(t) = \begin{cases} 0, & t < a, \\ M, & t \geq a. \end{cases}$$

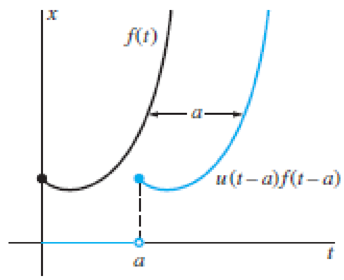
The Laplace transform of u_a is easily determined:

$$\begin{aligned} \mathcal{L}\{u_a(t)\} &= \int_0^\infty e^{-st} u_a(t) dt \\ &= \int_a^\infty e^{-st} dt \\ &= \lim_{b\to\infty} \int_a^b e^{-st} dt \\ &= \lim_{b\to\infty} \left(\frac{e^{-st}}{-s} \Big]_{t=a}^{t=b} \right) \\ &= \lim_{b\to\infty} \left(\frac{e^{-bs} - e^{-as}}{-s} \right) \\ &= \frac{e^{-as}}{s}, \qquad s > 0. \end{aligned} \tag{6.12}$$

For a given function f, defined for $t \geq 0$, we will often want to consider the related function g defined by

$$g(t) = u_a(t) f(t-a) = \begin{cases} 0, & t < a, \\ f(t-a), & t \geq a, \end{cases} \tag{6.13}$$

which represents a translation of f a distance a in the positive t direction as shown in the figure below.



We next present the following relation between the transform of $f(t)$ and that of its translation $u_a(t)f(t-a)$.

Theorem 6.9 *If* $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > c$*, then*

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s) \tag{6.14}$$

and

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = u_a(t)f(t-a), \qquad \text{for} \;\; s > a + c. \tag{6.15}$$

Thus the last theorem implies that $\mathcal{L}^{-1}\{e^{-as}F(s)\}$ is the function whose graph for $t \geq a$ is the translation by a units to the right of the graph of $f(t)$ for $t \geq 0$. Note that the part (if any) of the graph of $f(t)$ to the left of $t = 0$ is "cut off" and is not translated. In some applications the function $f(t)$ describes an incoming signal that starts arriving at time $t = 0$. Then $u_a(t)f(t-a)$ denotes a signal of the same "shape" but with a time delay of a, so it does not start arriving until time $t = a$.

In practice it is more common to be faced with the problem of computing the transform of a function expressed as $u_a(t)g(t)$ rather than $u_a(t)f(t-a)$. We can easily obtain from (6.14) that

$$\mathcal{L}\{u_a(t)g(t)\} = e^{-as}\mathcal{L}\{g(t+a)\}. \tag{6.16}$$

Example 6.17 Determine an inverse Laplace transform of the given function.

(1) $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\}$

To use the translation property (6.15), we first express $\frac{e^{-4s}}{s^3}$ as the product $e^{-as}F(s)$. For this purpose, we put $a = 4$ and $F(s) = \frac{1}{s^3}$. Thus,

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2.$$

It now follows from the translation property (6.15) that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\} &= u_4(t)f(t-4) = \frac{1}{2}\left(t-4\right)^2 u_4(t)\\ &= \begin{cases} 0, & t < 4,\\ \frac{1}{2}\left(t-4\right)^2, & t \geq 4.\end{cases}\end{aligned}$$

(2) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+9}\right\}$

We use the translation property (6.15) with $a = 2$ and $F(s) = \frac{1}{s^2+9}$. Since

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t,$$

we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+9}\right\} &= u_2(t)f(t-2) = \frac{\sin(3t-6)}{3}u_2(t)\\ &= \begin{cases} 0, & t < 2,\\ \frac{1}{3}\sin(3t-6), & t \geq 2.\end{cases}\end{aligned}$$

(3) $\mathcal{L}^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\}$

We use the translation property (6.15) with $a = 3$ and $F(s) = \frac{s}{s^2+4s+5}$. By completing the square in the denomenator,

$$\begin{aligned}\frac{s}{s^2+4s+5} &= \frac{s}{s^2+4s+(4-4)+5} = \frac{s}{(s^2+4s+4)+(5-4)}\\ &= \frac{s}{\left(s+2\right)^2+1} = \frac{s+2-2}{\left(s+2\right)^2+1}\\ &= \frac{s+2}{\left(s+2\right)^2+1} - \frac{2}{\left(s+2\right)^2+1}.\end{aligned}$$

Since

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} \\ &= e^{-2t}\cos t - 2e^{-2t}\sin t \\ &= e^{-2t}(\cos t - 2\sin t)\end{aligned}$$

we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\} &= u_3(t)f(t-3) \\ &= e^{-2(t-3)}\left[\cos(t-3) - 2\sin(t-3)\right]u_3(t).\end{aligned}$$

Example 6.18 Determine the Laplace transform of $t^2u_3(t)$.

Solution: To apply equation (6.16), we take $g(t) = t^2$ and $a = 3$. Then

$$g(t+a) = g(t+3) = (t+3)^2 = t^2 + 6t + 9.$$

Now the Laplace transform of $g(t+a)$ is

$$\mathcal{L}\{g(t+a)\} = \mathcal{L}\left\{t^2 + 6t + 9\right\} = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}.$$

So, by formula (6.16), we have

$$\mathcal{L}\left\{t^2u_3(t)\right\} = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

Example 6.19 Determine $\mathcal{L}\{(\sin t)\,u_\pi(t)\}$

Solution: Here $g(t) = \sin t$ and $a = \pi$. Hence,

$$g(t+a) = \sin(t+\pi) = -\sin t$$

and so the Laplace transform of $g(t+a)$ is

$$\mathcal{L}\{g(t+a)\} = -\mathcal{L}\{\sin t\} = -\frac{1}{s^2+1}.$$

Thus, from formula (6.16), we get

$$\mathcal{L}\{(\sin t)\,u_\pi(t)\} = -\frac{e^{-\pi s}}{s^2+1}.$$

Example 6.20 Find $\mathcal{L}\{f(t)\}$ if

$$f(t) = \begin{cases} 0, & t < 2, \\ t^2, & t \geq 2. \end{cases}$$

Solution: Before applying formula (6.16), we must first write $f(t)$ in the form $g(t)u_a(t)$. The function $f(t)$ can be expressed as

$$f(t) = t^2 u_3(t).$$

Here $g(t) = t^2$ and $a = 2$. But then

$$\mathcal{L}\{g(t+2)\} = \mathcal{L}\left\{(t+2)^2\right\} = \mathcal{L}\left\{t^2 + 4t + 4\right\} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

so now formula (6.16) yields

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^2 u_3(t)\right\} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).$$

6.5 Solving Initial Value Problems

We now discuss the application of Laplace transforms to solve a linear differential equation with constant coefficients, such as

$$ay'' + by' + cy = f(t), \tag{6.17}$$

with given initial conditions $y(0) = \alpha$ and $y'(0) = \beta$. By the linearity of the Laplace transformation, we can transform Eq. (6.17) by separately taking the Laplace transform of each term in the equation. The transformed equation is

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}. \tag{6.18}$$

The key to the method is the theorem that tells us how to express the transform of the derivative of a function in terms of the transform of the function itself,

$$\mathcal{L}\left\{y^{(n)}\right\}(s) = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f\}(s)$. For example,

$$\begin{aligned} \mathcal{L}\{y'\}(s) &= sY(s) - y(0), \\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - y(0)s - y'(0), \\ \mathcal{L}\{y'''\}(s) &= s^3Y(s) - y(0)s^2 - y'(0)s - y''(0). \end{aligned}$$

With the given initial conditions, Eq. (6.18) becomes

$$a\left(s^2Y(s) - y(0)s - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = F(s),$$

where $Y(s) = \mathcal{L}\{y(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$. Solving for $Y(s)$ yields

$$Y(s) = \frac{F(s)}{as^2 + bs + c} + \frac{(as + b)\,y(0) + ay'(0)}{as^2 + bs + c}.$$

Our remaining task is to compute the inverse transform of the rational function $Y(s)$,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

This can be done by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

As further illustrations of the technique of solving initial value problems bymeans of the Laplace transform and partial fraction expansions, consider the following examples.

Example 6.21 Solve the initial value problem

$$y'' - y' - 6y = 0; \qquad y(0) = 2, \qquad y'(0) = -1. \tag{6.19}$$

Solution: Let $F(s) = \mathcal{L}\{y(t)\}$. The differential equation in (6.19) is an identity between two functions of t. Hence equality holds for the Laplace transforms of these functions:

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\}.$$

Using the linearity property of $\mathcal{L}$ and that $\mathcal{L}\{0\} = 0$, we can write

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0, \tag{6.20}$$

From the formulas for the Laplace transform of higher-order derivatives and the initial conditions in (6.19), we find

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 2,$$
$$\mathcal{L}\{y''\}(s) = s^2Y(s) - y(0)s - y'(0) = s^2Y(s) - 2s + 1.$$

Substituting these expressions into (6.20) yields

$$s^2Y(s) - 2s + 1 - (sY(s) - 2) - 6Y(s) = 0,$$

which we quickly simplify to

$$\left(s^2 - s - 6\right)Y(s) = 2s - 3.$$

Thus,

$$Y(s) = \frac{2s-3}{s^2-s-6}. \tag{6.21}$$

Our remaining task is to compute the inverse transform of the rational function $Y(s)$. This can be done most easily by expanding the right side of Eq. (6.21) in partial fractions. Thus we write

$$Y(s) = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}.$$

Multiplication of both sides of this equation by $(s-3)(s+2)$ yields the identity

$$2s-3 = A(s+2) + B(s-3).$$

If we substitute $s = 3$, we find that $A = \frac{3}{5}$; substitution of $s = -2$ shows that $B = \frac{7}{5}$. Hence

$$Y(s) = \frac{\frac{3}{5}}{s-3} + \frac{\frac{7}{5}}{s+2},$$

and taking the inverse transform of this yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{Y(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\frac{3}{5}}{s-3} + \frac{\frac{7}{5}}{s+2}\right\} \\ &= \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}, \end{aligned}$$

as the solution to the initial value problem (6.19).

Example 6.22 Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \qquad y(0) = 2, \qquad y'(0) = 3. \tag{6.22}$$

Solution: Let $\mathcal{L}\left\{y(t)\right\} = Y(s)$. Taking Laplace transforms of both sides of the differential equation in (6.22) yields

$$\mathcal{L}\left\{y'' - 6y' + 5y\right\} = \mathcal{L}\left\{3e^{2t}\right\},$$

which we rewrite as

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \frac{3}{s-2}. \tag{6.23}$$

Then the initial conditions in (6.22) imply that

$$\begin{aligned}\mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 2,\\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - y(0)s - y'(0) = s^2Y(s) - 2s - 3.\end{aligned}$$

Substituting from the last two equations into (6.23) yields

$$s^2Y(s) - 2s - 3 - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$\left(s^2 - 6s + 5\right)Y(s) = \frac{3}{s-2} + 2s - 9,$$

so

$$(s-5)(s-1)Y(s) = \frac{(2s-9)(s-2)+3}{s-2},$$

and

$$Y(s) = \frac{(2s-9)(s-2)+3}{(s-1)\,(s-2)(s-5)}.$$

Using partial fractions we can write $Y(s)$ in the form

$$\frac{(2s-9)(s-2)+3}{(s-1)\,(s-2)(s-5)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-5}.$$

Multiplication of both sides of this equation by $(s-1)(s-2)(s-5)$ yields

$$(2s-9)(s-2)+3 = A(s-2)(s-5) + B(s-1)(s-5) + C(s-1)(s-2).$$

If we substitute $s = 1$, we find that $A = \frac{5}{2}$; substitution of $s = 2$ shows that $B = -1$; substitution of $s = 5$ gives that $C = \frac{1}{2}$. Hence

$$Y(s) = \frac{\frac{5}{2}}{s-1} - \frac{1}{s-2} + \frac{\frac{1}{2}}{s-5},$$

and taking the inverse transform of this yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} \\ &= \frac{5}{2}e^{t} - e^{2t} + \frac{1}{2}e^{5t}, \end{aligned}$$

which is the solution to the initial value problem (6.22).

Example 6.23 Solve the initial value problem

$$y'' + 4y = \sin 3t; \qquad y(0) = 0, \qquad y'(0) = 0. \tag{6.24}$$

Solution: Let $\mathcal{L}\{y(t)\} = Y(s)$. Applying the Laplace operator to the differential equation in (6.24) yields

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin 3t\}.$$

Because both initial values are zero, we get that

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - y(0)s - y'(0) = s^2Y(s).$$

We read the transform of $\sin 3t$ from Laplace table and thereby get the transformed equation

$$s^2Y(s) + 4Y(s) = \frac{3}{s^2+9}.$$

Thus,

$$Y(s) = \frac{3}{(s^2+4)(s^2+9)}.$$

By the method of partial fractions, there exist constants A, B, C and D such that

$$\frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

and multiplication of both sides of this equation by $\left(s^2+4\right)\left(s^2+9\right)$ yields the identity

$$\begin{aligned} 3 &= (As+B)\left(s^2+9\right) + (Cs+D)\left(s^2+4\right), \\ 3 &= (A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D) \end{aligned}$$

for all s. Then, comparing coefficients of like powers of s, we have

$$A + C = 0, \qquad B + D = 0, \qquad 9A + 4C = 0, \qquad 9B + 4D = 3,$$

which are readily solved for $A = 0, B = \frac{3}{5}, C = 0, D = -\frac{3}{5}$. Hence

$$Y(s) = \frac{3}{5}\frac{1}{s^2+4} - \frac{3}{5}\frac{1}{s^2+9}$$

and taking the inverse transform of this gives the solution

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \\ &= \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t. \end{aligned}$$

Example 6.24 A mass that weighs 32 lb is attached to the free end of a long, light spring that is stretched 1 ft by a force of 4 lb. The mass is initially at rest in its equilibrium position. Beginning at time $t = 0$ (seconds), an external force $f(t) = \cos 2t$ is applied to the mass, but at time $t = 2\pi$ this force is turned off (abruptly discontinued) and the mass is allowed to continue its motion unimpeded. Find the resulting position function $x(t)$ of the mass.

Solution: We need to solve the initial value problem

$$x'' + 4x = f(t); \qquad x(0) = 0, \qquad x'(0) = 0, \tag{6.25}$$

where $f(t)$ is the function

$$f(t) = \begin{cases} \cos 2t, & 0 \le t < 2\pi, \\ 0, & t \ge 2\pi. \end{cases}$$

We note first that

$$f(t) = \cos 2t - u_{2\pi}\cos 2t = \cos 2t - u_{2\pi}\cos 2\left(t - 2\pi\right),$$

because of the periodicity of the cosine function. Hence formula (6.14) gives

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{\cos 2t - u_{2\pi}\cos 2\left(t-2\pi\right)\} \\ &= \mathcal{L}\{\cos 2t\} - \mathcal{L}\{u_{2\pi}\cos 2\left(t-2\pi\right)\} \\ &= \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-2\pi s}. \end{aligned}$$

Applying the Laplace transforms to the equation in (6.25) gives the transformed equation

$$s^2X(s) + 4X(s) = \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-2\pi s},$$

so

$$X(s) = \frac{s}{\left(s^2+4\right)^2}\frac{s}{\left(s^2+4\right)^2}e^{-2\pi s}.$$

Because

$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+4\right)^2}\right\} = \frac{1}{4}t\sin 2t$$

by formula (6.10), it follows from the translation property (6.15) that

$$\begin{aligned} x(t) &= \frac{1}{4}t\sin 2t - u_{2\pi}\left[\frac{1}{4}(t-2\pi)\sin 2(t-2\pi)\right] \\ &= \frac{1}{4}\left[t-(t-2\pi)\,u_{2\pi}(t)\right]\sin 2t \\ &= \begin{cases} \frac{1}{4}t\sin 2t, & 0 \le t < 2\pi, \\ \frac{1}{2}\pi\sin 2t, & t \ge 2\pi. \end{cases} \end{aligned}$$